Internet Appendix for Default Risk and the Pricing of U.S. Sovereign Bonds[1]

A TIPS, Nominal Treasuries, and CDS

Our empirical analysis focuses on four financial instruments: inflation-indexed Treasury notes (TIPS), nominal Treasury notes, credit default swaps on the U.S. Treasury (CDS), and inflation-indexed swaps (ILS). In this section we discuss the zero-coupon arbitrage in ILS, nominal, and inflation-protected Treasuries, an analogue to the coupon-based strategy detailed in Fleckenstein, Longstaff and Lustig (2014). We also discuss institutional features of these securities as well as CDS, and how these might relate to the profitability of the arbitrage strategy.

The two primary instruments of debt issuance for the U.S. government are cash-denominated Treasuries (nominal) and *Treasury inflation-protected securities* (TIPS). Nominal zero-coupon bonds pay their nominal face value to the bond-holder at maturity. In contrast, zero-coupon TIPS holders earn the inflation-adjusted face value of the bond at maturity. Since cumulative inflation tends to be positive, TIPS tend to trade at a premium compared to nominal bonds. For both nominal bonds and TIPS, yields at issuance are determined through an auction process involving numerous market participants. According to Treasury Direct, as of April, 2020, the total principal value of Treasury securities outstanding is $18,104 billion, of which $1,493 billion, or 8% are TIPS. The dollar amount of TIPS outstanding is comparable in magnitude to each of the respective markets for asset-backed securities, federal agency securities, and U.S. money market instruments.[2]

The TIPS inflation adjustment is computed using the *seasonally non-adjusted consumer price index* for all urban consumers in the U.S. (CPI-U). CPI data is published monthly by the Bureau of Labor Statistics with a lag of about one and a half months, making the realized inflation unavailable when TIPS mature. TIPS payments thus include an *indexation lag* — the index used to determine their cashflows is a linear interpolation of CPI-U observed between two and three

[1] Dittmar, Robert, Alex Hsu, and Guillaume Roussellet, Internet Appendix to "Default Risk and the Pricing of U.S. Sovereign Bonds," Journal of Finance [DOI STRING]. Please note: Wiley is not responsible for the content or functionality of any supporting information supplied by the authors. Any queries (other than missing material) should be directed to the authors of the article.

[2] https://www.sifma.org/resources/research/fixed-income-chart/

months before. The inflation-adjusted principal paid back at maturity is calculated by multiplying the face value of the bond by the cumulative *index ratio*. TIPS embed a *deflation floor*, such that they return the full face value even if cumulative inflation realized over the bond lifetime is negative.[3]

Despite the indexation lag, it would be difficult for the U.S. government to inflate away outstanding TIPS. Technically, it would be possible for the sovereign to resort to seignoriage to pay back maturing TIPS and current coupon payments without realizing the consequence of increased inflation. However, the inflation adjustment will materially impact any remaining outstanding TIPS, increasing the future interest payments of the government. Should the U.S. government refuse to honor the TIPS indexation, this would likely trigger a credit event and force the payoff of U.S. CDS contracts (see below). In case of default, nominal bonds and TIPS have the same level of seniority.

Leaving aside the embedded deflation floor, TIPS can be theoretically replicated by combining nominal bonds and inflation-linked swaps (ILS), as shown in Fleckenstein, Longstaff and Lustig (2014). ILS allow for the buyer to earn cumulative inflation in exchange for a fixed rate, relative to the notional agreed upon at inception. Inflation swaps are costless to write, and they are typically zero-coupon. As of April 2012, the average daily brokered inflation swap activity was estimated to be $350 million, concentrated around the 10-year maturity. Importantly, despite a low trading frequency averaging about 2.2 contracts per day, the market for inflation swaps appears fairly liquid, with bid-ask spreads from proprietary data averaging below 3 basis points.[4] Keeping with the standard for swap contracts, ILS are collateralized, thus subject to minimal counterparty risk. In the remainder of the paper, we will assume that ILS are virtually risk-free.

In a frictionless economy, for a given maturity n, no arbitrage implies that the zero-coupon ILS rate is equal to the spread between the nominal and TIPS zero-coupon yields, called *breakeven inflation rate* (BEI):

$$\mathrm{ILS}_t^{(n)} = R_t^{(n)} - R_t^{(n)^*} = \mathrm{BEI}_t^{(n)} \,. \tag{12}$$

This measure is the zero-coupon equivalent of Fleckenstein, Longstaff and Lustig (2014), who

[3]We consider zero coupon bonds in this study. Note however that most of nominal bonds and TIPS issued by the U.S. sovereign are coupon bonds paying on a semi-annual basis, but TIPS are only issued in terms of five, ten, twenty, or thirty years. For TIPS coupon payments, the coupon rate is fixed and paid on the inflation adjusted principal. For coupon payments, there is no deflation floor and the inflation-adjustment is computed using the index ratio realized over the last 6 months.

[4]https://libertystreeteconomics.newyorkfed.org/2013/04/how-liquid-is-the-inflation-swap-market.html and *JPMorgan Investment Insight: Inflation Derivatives.*

show that the cash flows of any traded nominal Treasury bond can be replicated by a portfolio of TIPS, U.S. Treasury STRIPS, and inflation swaps. We equivalently call this spread ILSBEI, *mispricing*, or *hedged breakeven*.

In practice, researchers have observed large deviations from this no-arbitrage relationship over the maturity spectrum. Figures 1 and 2 present the five years to maturity series of ILS and BEI and the term structure of the spread between inflation swap rates and zero-coupon BEI, respectively. These deviations from the no-arbitrage relationship are quite persistent, and average between 30 and 36 basis points depending on the maturity. In the midst of the crisis, they reached more than 200 basis points.

Most of this apparent mispricing has been previously attributed to the low liquidity of TIPS relative to nominal bonds and ILS or to slow-moving capital (see e.g. D'Amico, Kim and Wei (2018) or Fleckenstein, Longstaff and Lustig (2014)). Campbell, Shiller and Viceira (2009) suggest the premium is related to the cost of supplying inflation protection and is typical under normal market conditions. Inflation swaps, Treasuries, and TIPS all trade over-the-counter and may be subject to varying liquidity risk or counterparty credit risk in the case of ILS. We argue that the ILSBEI spread is also significantly related to the risk of default of the U.S. sovereign that we associate with U.S. CDS. In our analysis, we control for all these potential confounding factors in the analysis, and abstract away from the embedded deflation floor in TIPS and the tax-related issues. We note that the deflation floor drives the price of TIPS upward, making the observed TIPS yield lower than the one used in the no-arbitrage argument. This would lead us to *underestimate* the ILSBEI spread, thus the size of the potential mispricing.

Credit default swaps (CDS) are OTC instruments designed to protect bond investors from a contingent *credit event* of the issuing entity. In practice, a bond investor (*protection buyer*) entering a CDS agrees to pay a fixed premium, typically called the *CDS spread*, on a regular basis to the *protection seller*, her counterparty. In case of a credit event, the contract terminates and the seller has to deliver the *loss given default* (LGD) realized on the bond to the buyer, making her earn the entire face value of the bond upon default. As is standard for swap contracts, the premium is indexed on a notional amount agreed upon at inception and is set such that the original cost of issuance is zero. While not free from counterparty credit risk, CDS are typically collateralized.

The International Swaps and Derivatives Association (ISDA) provides legal details that define the triggers for the termination of CDS, which type of obligations are considered, and how the LGD and repayment operates depending on the underlying bond issuer (see ISDA (2003, 2014)). In the case of the United States Treasury, a credit event is observed whenever the government either *(i)* fails to repay, *(ii)* repudiates or imposes a moratorium, or *(iii)* restructures any of its borrowed money. This includes in particular any Treasury Bill, Bond or Note, whether nominal or indexed. In our empirical analysis, we identify default with the conditions for which CDS protection are triggered.

In the case of a credit event, the LGD is determined through an auction addressed to CDS dealer banks. Participating banks typically submit a bid and ask quote on a $100 face-value bond of the reference entity, and the cross-section of bid-asks is used to determine the final price of the bond, typically below par (see Augustin et al. (2014)).[5]

Settlement of the CDS contract can be completed either through cash or physical delivery. In the former case, the protection seller delivers a payment equal to the LGD as determined by the auction, multiplied by the notional of the CDS. In the latter case, the protection seller pays the entire notional to the buyer in exchange for an equivalent principal amount of reference bonds. If these bonds have the exact same characteristics as those auctioned, the two deliveries would be equivalent. However, the protection buyer can choose to exchange *any of her reference bonds* with maturity below 30 years and above the maturity of the CDS contract. This essentially embeds a *cheapest-to-deliver* (CtD) option to the buyer's position, who will likely deliver the lowest dollar price reference obligation available.[6]

U.S. CDS contracts fall under the "Big Bang Protocol" established by ISDA in 2009. In the aftermath of the financial crisis, as the primary industry body overseeing swaps and derivative trading, ISDA pushed swap market participants to adopt the new protocol in an effort to standardize over-the-counter contract parameters.[7] A number of the implemented changes are worth highlighting. First, coupon payments on each contract are fixed at either 100 (investment grade) or 500 basis points (non-investment grade). As a result, there is typically a payment to be made at the initiation of the contract to ensure that the present values of expected cash flows are

[5]The final price of the bond resulting from this auction is published by CreditEx (`http://www.creditfixings.com/CreditEventAuctions/results.jsp`).

[6]In the context of the Greek crisis, CDS contracts and the associated auction mechanism played a minor role in the resturcturing process. As highlighted by Zettelmeyer, Trebesch and Gulati (2013), the credit event was triggered only after the preemptive debt restucturing. Therefore, the CDS auction took place after the bond exchange, and the resulting auction price fell in place with the new bond price in the secondary market. To be certain, CDS coverage of Greek sovereign debt was very low, at less than 2%. One would not expect the outcome of the bond auction to dictate terms of the restructuring.

[7]BIS Quarterly Review, December 2010. "The Big Bang in the CDS Market"

equal between the buyer's and seller's legs. A second important change stemming from the protocol is the hardwiring of the auction process following credit events such that all protection buyers obtain fair cash payments from protection sellers. Third, the protocol further stipulates the creation of Determinations Committees for determining whether a credit or succession event has occurred in order to reduce disputes between counterparties in case there is ambiguity.

Market participants in the sovereign CDS market include security dealers, banks and other financial institutions, and hedge funds (see e.g. Augustin (2018)). There is evidence that sovereign CDS contracts are used in both a hedging and speculative context. For contracts specifically written on the U.S. sovereign, focusing on the most liquid contracts with five years to maturity, price data from Markit shows there is very little pricing movement before the financial crisis of 2008. The premium spiked in 2009, at the height of the crisis, to about 100 basis points and has remained elevated afterward between 20 to 40 basis points.

Chernov, Schmid and Schneider (2020) provide a detailed discussion on the determinants of U.S. sovereign CDS spread beyond credit risk. For instance, the majority of U.S. CDS contracts are denominated in euros, and there is a small foreign exchange premium embedded in the spread. U.S. dollar denominated contracts did not start trading until August 2010 and volumes are thin relative to euro contracts. Additionally, there is uncertainty in the cheapest-to-deliver option due to the bond auction protocol conditional on default occurring. Lastly, the U.S. CDS spread should contain a liquidity premium component due to the relative scarcity of the instrument compared to other sovereign CDS contracts. A combination of these factors contribute to the U.S. sovereign CDS premium.

In the context of our project, we use the U.S. sovereign CDS premium as the indicator of the market's beliefs about default risk. We recognize that non-credit risk-related factors may influence the pricing of CDS. In the next section, we detail our approach to this issue and present several robustness tests in the Appendix to rule out the possibility these non-credit risk-related factors can simultaneously generate differential prices in U.S. sovereign bonds.

B Proofs in the simple model

B.1 Model

We assume the following specification for the pricing kernel and inflation dynamics:

$$\begin{aligned} \log M_{t+1}^* &= \overline{M} + \Lambda_\delta \cdot \mathbb{1}\left\{\delta_{t+1}^{(c)} > 0\right\}, \\ \pi_{t+1} &= \kappa_0 + \kappa_y \cdot \lambda_t + \kappa_\delta \cdot \mathbb{1}\left\{\delta_{t+1}^{(c)} > 0\right\}, \end{aligned} \tag{13}$$

where $\delta_{t+1}^{(c)}$ is a non-negative process with conditional jump probability λ_t.

B.2 One-period pricing

Riskless securities: We calculate the price of a riskless real bond first.

$$\begin{aligned} D_t^{(1)^*} &= \mathbb{E}_t\left(M_{t+1}^*\right) \\ &= \lambda_t \cdot \mathbb{E}\left(M_{t+1}^* \,\middle|\, \delta_{t+1}^{(c)} > 0\right) + (1-\lambda_t) \cdot \mathbb{E}\left(M_{t+1}^* \,\middle|\, \delta_{t+1}^{(c)} = 0\right) \\ &= \lambda_t \cdot e^{\overline{M}+\Lambda_\delta} + (1-\lambda_t) \cdot e^{\overline{M}} \\ &= e^{\overline{M}} \cdot \left[1 + \lambda_t\left(e^{\Lambda_\delta} - 1\right)\right]. \end{aligned} \tag{14}$$

This means the one-period yield is approximately given by:

$$r_t^{(1)^*} \underset{\lambda_t \to 0}{\sim} -\overline{M} - \lambda_t\left(e^{\Lambda_\delta} - 1\right) \sim -\overline{M} - \Lambda_\delta \lambda_t \tag{15}$$

In the same spirit, nominal bond prices can be obtained as:

$$\begin{aligned} D_t^{(1)} &= \mathbb{E}_t\left(M_{t+1}^* e^{-\pi_{t+1}}\right) \\ &= \lambda_t \cdot \mathbb{E}\left(M_{t+1}^* e^{-\pi_{t+1}} \,\middle|\, \delta_{t+1}^{(c)} > 0\right) + (1-\lambda_t) \cdot \mathbb{E}\left(M_{t+1}^* e^{-\pi_{t+1}} \,\middle|\, \delta_{t+1}^{(c)} = 0\right) \\ &= \lambda_t \cdot e^{\overline{M}-\kappa_0-\kappa_y\lambda_t+\Lambda_\delta-\kappa_\delta} + (1-\lambda_t) \cdot e^{\overline{M}-\kappa_0-\kappa_y\lambda_t} \\ &= e^{\overline{M}-\kappa_0-\kappa_y\lambda_t}\left[1 + \lambda_t\left(e^{\Lambda_\delta-\kappa_\delta} - 1\right)\right] \end{aligned} \tag{16}$$

The corresponding yield is given by:

$$r_t^{(1)} \underset{\lambda_t \to 0}{\sim} -\overline{M} + \kappa_0 + \lambda_t \left[\kappa_y - \left(e^{\Lambda_\delta - \kappa_\delta} - 1\right)\right] \sim -\overline{M} + \kappa_0 + \left(\kappa_y + \kappa_\delta - \Lambda_\delta\right)\lambda_t \tag{17}$$

Thus we can obtain ILS rates as:

$$\mathrm{ILS}_t^{(1)} \underset{\lambda_t \to 0}{\sim} \kappa_0 + \lambda_t \left[\kappa_y + e^{\Lambda_\delta}\left(1 - e^{-\kappa_\delta}\right)\right] \sim \kappa_0 + \left(\kappa_y + \kappa_\delta\right)\lambda_t \,. \tag{18}$$

Defaultable treasuries: Let us turn now to risky securities. Let us assume the loss given default of TIPS is given by LGD*. The price of a risky real treasury is given by:

$$\begin{aligned}
B_t^{(1)^*} &= \lambda_t \cdot (1 - \mathrm{LGD}^*) \cdot \mathbb{E}\left(M_{t+1}^* \,\middle|\, \delta_{t+1}^{(c)} > 0\right) + (1 - \lambda_t) \cdot \mathbb{E}\left(M_{t+1}^* \,\middle|\, \delta_{t+1}^{(c)} = 0\right) \\
&= D_t^{(1)^*} - \lambda_t \cdot \mathrm{LGD}^* \cdot \mathbb{E}\left(M_{t+1}^* \,\middle|\, \delta_{t+1}^{(c)} > 0\right) \\
&= D_t^{(1)^*}\left[1 - \lambda_t \cdot \mathrm{LGD}^* \cdot \frac{\mathbb{E}\left(M_{t+1}^* \,\middle|\, \delta_{t+1}^{(c)} > 0\right)}{D_t^{(1)^*}}\right] \\
&= D_t^{(1)^*}\left[1 - \lambda_t \cdot \mathrm{LGD}^* \cdot \frac{e^{\Lambda_\delta}}{1 + \lambda_t\left(e^{\Lambda_\delta} - 1\right)}\right] \\
&= e^{\overline{M}} \cdot \left[1 + \lambda_t\left(e^{\Lambda_\delta} - 1\right) - \lambda_t \mathrm{LGD}^* e^{\Lambda_\delta}\right] \\
&= e^{\overline{M}} \cdot \left[1 + \lambda_t\left(\left[1 - \mathrm{LGD}^*\right] e^{\Lambda_\delta} - 1\right)\right]
\end{aligned} \tag{19}$$

Again, the yield is given by:

$$R_t^{(1)^*} \underset{\lambda_t \to 0}{\sim} -\overline{M} - \lambda_t\left(\left[1 - \mathrm{LGD}^*\right] e^{\Lambda_\delta} - 1\right) \sim -\overline{M} - \Lambda_\delta \lambda_t + \left(1 + \Lambda_\delta\right)\mathrm{LGD}^* \cdot \lambda_t \tag{20}$$

The real credit spread is then given by

$$R_t^{(1)^*} - r_t^{(1)^*} \sim \lambda_t \cdot e^{\Lambda_\delta} \cdot \mathrm{LGD}^* \tag{21}$$

Similarly, the price of a risky nominal treasury whose loss given default is denoted by LGD is given by:

$$
\begin{aligned}
B_t^{(1)} &= \lambda_t \cdot (1 - 1 - \text{LGD}) \cdot \mathbb{E}\left(M_{t+1}^* e^{-\pi_{t+1}} \mid \delta_{t+1}^{(c)} > 0 \right) + (1 - \lambda_t) \cdot \mathbb{E}\left(M_{t+1}^* e^{-\pi_{t+1}} \mid \delta_{t+1}^{(c)} = 0 \right) \\
&= D_t^{(1)} - \lambda_t \cdot \text{LGD} \cdot \mathbb{E}\left(M_{t+1}^* e^{-\pi_{t+1}} \mid \delta_{t+1}^{(c)} > 0 \right) \\
&= D_t^{(1)} \left[1 - \lambda_t \cdot \text{LGD} \cdot \frac{\mathbb{E}\left(M_{t+1}^* e^{-\pi_{t+1}} \mid \delta_{t+1}^{(c)} > 0 \right)}{D_t^{(1)}} \right] \\
&= D_t^{(1)} \left[1 - \lambda_t \cdot \text{LGD} \cdot \frac{e^{\Lambda_\delta - \kappa_\delta}}{1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1 \right)} \right] \\
&= e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t + \frac{\sigma_m^2 + \sigma_\pi^2}{2}} \left[1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1 \right) \right] \left[1 - \lambda_t \cdot \text{LGD} \cdot \frac{e^{\Lambda_\delta - \kappa_\delta}}{1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1 \right)} \right] \\
&= e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t} \left[1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1 \right) - \lambda_t \cdot \text{LGD} \cdot e^{\Lambda_\delta - \kappa_\delta} \right] \\
&= e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t} \left[1 + \lambda_t \left([1 - \text{LGD}] \cdot e^{\Lambda_\delta - \kappa_\delta} - 1 \right) \right] . \qquad (22)
\end{aligned}
$$

The corresponding yield is given by:

$$
R_t^{(1)} \underset{\lambda_t \to 0}{\sim} -\overline{M} + \kappa_0 + \lambda_t \left[\kappa_y - \left([1 - \text{LGD}] \cdot e^{\Lambda_\delta - \kappa_\delta} - 1 \right) \right] \sim r_t + (1 + \Lambda_\delta - \kappa_\delta) \cdot \text{LGD} \cdot \lambda_t \qquad (23)
$$

and the nominal credit spread is given by:

$$
R_t^{(1)} - r_t^{(1)} = \lambda_t \cdot e^{\Lambda_\delta - \kappa_\delta} \cdot \text{LGD} \qquad (24)
$$

The BEI and ILS-BEI spreads are respectively given by:

$$
\begin{aligned}
\text{BEI}_t^{(1)} &\sim \kappa_0 + \lambda_t \left[\kappa_y + e^{\Lambda_\delta} \left(1 - \text{LGD}^* - [1 - \text{LGD}] \cdot e^{-\kappa_\delta} \right) \right] \qquad (25) \\
\text{ILSBEI}_t^{(1)} &\sim \lambda_t \cdot e^{\Lambda_\delta} \left[\text{LGD}^* - e^{-\kappa_\delta} \text{LGD} \right] \qquad (26)
\end{aligned}
$$

B.3 Multi-period pricing with independence

To keep things simple, we assume that the default probability is fixed through time and that we draw the default process **independently every period**. We relax this assumption later. Following our assumption, it is easy to see that the price of bonds is fixed through time as long as the entity is alive. Thus, it is easy to obtain the complete pricing recursions.

Riskless bonds: Let us consider inflation-indexed bonds first. We have:

$$\begin{aligned} D_t^{(n)^*} &= \left(\lambda_t \cdot e^{\Lambda_\delta} + 1 - \lambda_t\right) \mathbb{E}_t \left[e^{\overline{M}} D^{(n-1)^*} \right] \\ &= e^{n\overline{M}} \cdot \left[1 + \lambda_t \left(e^{\Lambda_\delta} - 1\right)\right]^n \end{aligned} \tag{27}$$

therefore:

$$r_t^{(n)^*} = -\overline{M} - \log\left[1 + \lambda_t \left(e^{\Lambda_\delta} - 1\right)\right] \tag{28}$$

Similarly, for nominal riskless bonds, we have:

$$\begin{aligned} D_t^{(n)} &= \left(\lambda_t \cdot e^{\Lambda_\delta - \kappa_\delta} + 1 - \lambda_t\right) \mathbb{E}_t \left[e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t} D^{(n-1)} \right] \\ &= e^{n\left(\overline{M} - \kappa_0 - \kappa_y \lambda_t\right)} \cdot \left[1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1\right)\right]^n . \end{aligned} \tag{29}$$

therefore:

$$r_t^{(n)} = -\overline{M} + \kappa_0 + \kappa_y \lambda_t - \log\left[1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1\right)\right] \tag{30}$$

The ILS is then given by:

$$\begin{aligned} \mathrm{ILS}_t^{(n)} &= \kappa_0 + \kappa_y \lambda_t + \log\left[\frac{1 + \lambda_t \left(e^{\Lambda_\delta} - 1\right)}{1 + \lambda_t \left(e^{\Lambda_\delta - \kappa_\delta} - 1\right)}\right] \\ &\sim \kappa_0 + \kappa_y \lambda_t + \lambda_t \cdot e^{\Lambda_\delta} \left(1 - e^{-\kappa_\delta}\right) \end{aligned} \tag{31}$$

Risky bonds: Using the same logic, risky bond yields can be obtained as follows.

$$\begin{aligned} B_t^{(n)^*} &= \left(\lambda_t \cdot e^{\Lambda_\delta} \left(1 - \mathrm{LGD}^*\right) + 1 - \lambda_t\right) \mathbb{E}_t \left[e^{\overline{M}} B^{(n-1)^*} \right] \\ &= e^{n\overline{M}} \cdot \left[1 + \lambda_t \cdot \left(e^{\Lambda_\delta} \left(1 - \mathrm{LGD}^*\right) - 1\right)\right]^n . \end{aligned} \tag{32}$$

Thus:

$$R_t^{(n)^*} = -\overline{M} - \log\left[1 + \lambda_t \cdot \left(e^{\Lambda_\delta} \left(1 - \mathrm{LGD}^*\right) - 1\right)\right] \tag{33}$$

Similarly:

$$\begin{aligned} B_t^{(n)} &= \left(\lambda_t \cdot e^{\Lambda_\delta - \kappa_\delta} \left(1 - \mathrm{LGD}\right) + 1 - \lambda_t\right) \mathbb{E}_t \left[e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t} B^{(n-1)} \right] \\ &= e^{n\left(\overline{M} - \kappa_0 - \kappa_y \lambda_t\right)} \cdot \left[1 + \lambda_t \left(\left(1 - \mathrm{LGD}\right) e^{\Lambda_\delta - \kappa_\delta} - 1\right)\right]^n . \end{aligned} \tag{34}$$

therefore:

$$R_t^{(n)} = -\overline{M} + \kappa_0 + \kappa_y \lambda_t - \log\left[1 + \lambda_t \left((1 - \text{LGD})\, e^{\Lambda_\delta - \kappa_\delta} - 1\right)\right] \tag{35}$$

and the BEI and ILSBEI spreads are given by:

$$\begin{aligned}
\text{BEI}_t^{(n)} &= \kappa_0 + \kappa_y \lambda_t + \log\left[\frac{1 + \lambda_t \left((1 - \text{LGD}^*) \cdot e^{\Lambda_\delta} - 1\right)}{1 + \lambda_t \left((1 - \text{LGD}) \cdot e^{\Lambda_\delta - \kappa_\delta} - 1\right)}\right] \\
&\sim \kappa_0 + \kappa_y \lambda_t + \lambda_t \cdot e^{\Lambda_\delta} \left(1 - \text{LGD}^* - (1 - \text{LGD}) \cdot e^{-\kappa_\delta}\right)
\end{aligned} \tag{36}$$

and:

$$\text{ILSBEI}_t^{(n)} \sim \lambda_t \cdot e^{\Lambda_\delta} \left[\text{LGD}^* - e^{-\kappa_\delta} \text{LGD}\right] \tag{37}$$

B.4 Absorbing Default Process

We now consider that the effects of a default can be permanent. In other words, the conditional probability λ_t given a default occurring today is equal to 1. Notice that this assumption does not modify the one-period yields, but only affects the multi-period ones.

Riskless yields: We focus on inflation-indexed bonds first. We have a tree of outcomes. With probability $(1 - \lambda_t)^n$, the sovereign stays alive for n periods. Otherwise the sovereign default exactly at $t + k + 1$ with probability $(1 - \lambda_t)^k \lambda_t$. In this case, since the default state is absorbing, both the SDF and the inflation rate jump by Λ_δ and κ_δ, respectively, and for the remaining $n - k$ period. Putting all together we have:

$$\begin{aligned}
D_t^{(n)^*} &= (1 - \lambda_t)^n\, e^{n\overline{M}} + \sum_{k=0}^{n-1} (1 - \lambda_t)^k\, \lambda_t e^{n\overline{M} + \Lambda_\delta (n-k)} \\
&= e^{n\overline{M}} \left[(1 - \lambda_t)^n + \lambda_t e^{n\Lambda_\delta} \sum_{k=0}^{n-1} (1 - \lambda_t)^k\, e^{-k\Lambda_\delta}\right] \\
&= e^{n\overline{M}} \left[(1 - \lambda_t)^n + \lambda_t e^{n\Lambda_\delta} \frac{1 - (1 - \lambda_t)^n\, e^{-n\Lambda_\delta}}{1 - (1 - \lambda_t)\, e^{-\Lambda_\delta}}\right] \\
&= e^{n\overline{M}} \left[(1 - \lambda_t)^n \left(1 - \frac{\lambda_t}{1 - (1 - \lambda_t)\, e^{-\Lambda_\delta}}\right) + \frac{e^{n\Lambda_\delta} \lambda_t}{1 - (1 - \lambda_t)\, e^{-\Lambda_\delta}}\right] \\
&= e^{n\overline{M}} \frac{(1 - \lambda_t)^{n+1} \left(1 - e^{-\Lambda_\delta}\right) + e^{n\Lambda_\delta} \lambda_t}{1 - (1 - \lambda_t)\, e^{-\Lambda_\delta}}
\end{aligned} \tag{38}$$

and, for nominal bonds:

$$\begin{aligned} D_t^{(n)} &= (1-\lambda_t)^n \, e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)} + \sum_{k=0}^{n-1} (1-\lambda_t)^k \, \lambda_t e^{n\left(\overline{M}-\kappa_0\right)-n\kappa_y\lambda_t-(n-k-1)\kappa_y(1-\lambda_t)+(n-k)(\Lambda_\delta-\kappa_\delta)} \\ &= e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)} \left[(1-\lambda_t)^n + \lambda_t e^{n(\Lambda_\delta-\kappa_\delta)-(n-1)\kappa_y(1-\lambda_t)} \sum_{k=0}^{n-1} (1-\lambda_t)^k \, e^{k\kappa_y(1-\lambda_t)-k(\Lambda_\delta-\kappa_\delta)} \right] \\ &= e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)} \left[(1-\lambda_t)^n + \lambda_t e^{n(\Lambda_\delta-\kappa_\delta)-(n-1)\kappa_y(1-\lambda_t)} \frac{1-(1-\lambda_t)^n \, e^{n\kappa_y(1-\lambda_t)-n(\Lambda_\delta-\kappa_\delta)}}{1-(1-\lambda_t)\, e^{\kappa_y(1-\lambda_t)-(\Lambda_\delta-\kappa_\delta)}} \right] \\ &= e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)} \left[(1-\lambda_t)^n + \lambda_t \cdot e^{\kappa_y(1-\lambda_t)} \frac{e^{n(\Lambda_\delta-\kappa_\delta-\kappa_y(1-\lambda_t))}-(1-\lambda_t)^n}{1-(1-\lambda_t)\, e^{-(\Lambda_\delta-\kappa_\delta-\kappa_y(1-\lambda_t))}} \right] \end{aligned} \tag{39}$$

Defaultable bonds: We use recursive computations for defaultable bonds. Starting with TIPS, we have:

$$\begin{aligned} B_t^{(n)^*} &= \lambda_t \cdot \mathbb{E}_t\left(M_{t+1}^* \cdot (1-\mathrm{LGD}^*) \cdot B_{t+1}^{(n-1)^*} \,\middle|\, \delta_{t+1}^{(c)} > 0\right) + (1-\lambda_t)\,\mathbb{E}_t\left(M_{t+1}^* B_{t+1}^{(n-1)^*} \,\middle|\, \delta_{t+1}^{(c)} = 0\right) \\ &= e^{\overline{M}}\left[\lambda_t \cdot (1-\mathrm{LGD}^*) \cdot e^{\Lambda_\delta} \cdot \mathbb{E}_t\left(B_{t+1}^{(n-1)^*} \,\middle|\, \delta_{t+1}^{(c)} > 0\right) + (1-\lambda_t)\,\mathbb{E}_t\left(B_{t+1}^{(n-1)^*} \,\middle|\, \delta_{t+1}^{(c)} = 0\right)\right] \end{aligned} \tag{40}$$

Notice that the only source of time variation in the price is potentially λ_{t+1}, which can take 2 values. Conditionally on whether default has happened at $t+1$ the future price is known. Thus, we can write $B_{t+1}^{(n-1)^*} =: f_{n-1}^*(\lambda_{t+1})$. We obtain:

$$B_t^{(n)^*} = f_n^*(\lambda_t) = e^{\overline{M}}\left[\lambda_t \cdot (1-\mathrm{LGD}^*) \cdot e^{\Lambda_\delta} \cdot f_{n-1}^*(1) + (1-\lambda_t)\, f_{n-1}^*(\lambda_t)\right] \tag{41}$$

starting from $f_0^*(\lambda_t) = 1$. Since our model is a binomial tree with absorbing states, each price will be of the form:

$$f_n^*(\lambda_t) = e^{n\overline{M}}\left[\lambda_t \sum_{k=0}^{n-1} a_{n,k}^* \,(1-\lambda_t)^k + a_{n,n}^*\,(1-\lambda_t)^n\right]$$

We thus obtain the following recursion:

$$\begin{aligned} a_{n,k}^* &= a_{n-1,k-1}^* \quad \text{for} \quad k > 0 \\ a_{n,0}^* &= a_{n-1,0}^* \cdot (1-\mathrm{LGD}^*) \cdot e^{\Lambda_\delta} \end{aligned} \tag{42}$$

Starting from initial conditions $a_{0,0}^* = 1$, we obtain:

$$a_{n,k}^* = a_{n-1,k-1}^* = \ldots = a_{n-k,0}^* = (1-\mathrm{LGD}^*)^{n-k} \cdot e^{(n-k)\Lambda_\delta} \quad \text{for all} \quad k \leqslant n \tag{43}$$

Thus, we obtain:

$$B_t^{(n)^*} = e^{n\overline{M}} \left[\lambda_t \sum_{k=0}^{n-1} (1 - \text{LGD}^*)^{n-k} \cdot e^{(n-k)\Lambda_\delta} (1-\lambda_t)^k + (1-\lambda_t)^n \right] \tag{44}$$

Using the same manipulation as for riskless bonds, we obtain:

$$B_t^{(n)^*} = e^{n\overline{M}} \frac{(1-\lambda_t)^{n+1} \left(1 - (1-\text{LGD}^*)^{-1} \cdot e^{-\Lambda_\delta}\right) + (1-\text{LGD}^*)^n \cdot e^{n\Lambda_\delta} \lambda_t}{1 - (1-\lambda_t)(1-\text{LGD}^*)^{-1} \cdot e^{-\Lambda_\delta}} \tag{45}$$

We can also derive nominal bond prices in the same way.

$$\begin{aligned} B_t^{(n)} &= \lambda_t \cdot \mathbb{E}_t \left(M_{t+1}^* e^{-\pi_{t+1}} \cdot (1-\text{LGD}) \cdot B_{t+1}^{(n-1)} \,\middle|\, \delta_{t+1}^{(c)} > 0 \right) + (1-\lambda_t) \mathbb{E}_t \left(M_{t+1}^* e^{-\pi_{t+1}} B_{t+1}^{(n-1)} \,\middle|\, \delta_{t+1}^{(c)} = 0 \right) \\ &= e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t} \left[\lambda_t \cdot (1-\text{LGD}) \cdot e^{\Lambda_\delta - \kappa_\delta} \cdot \mathbb{E}_t \left(B_{t+1}^{(n-1)} \,\middle|\, \delta_{t+1}^{(c)} > 0 \right) + (1-\lambda_t) \mathbb{E}_t \left(B_{t+1}^{(n-1)} \,\middle|\, \delta_{t+1}^{(c)} = 0 \right) \right] \end{aligned}$$

Again, denoting by $B_t^{(n)} =: f_n(\lambda_t)$, we have

$$B_t^{(n)} = e^{\overline{M} - \kappa_0 - \kappa_y \lambda_t} \left[\lambda_t \cdot (1-\text{LGD}) \cdot e^{\Lambda_\delta - \kappa_\delta} \cdot f_{n-1}(1) + (1-\lambda_t) f_{n-1}(\lambda_t) \right] \tag{46}$$

where

$$f_n(\lambda_t) = e^{n\left(\overline{M} - \kappa_0 - \kappa_y \lambda_t\right)} \left[\lambda_t \sum_{k=0}^{n-1} a_{n,k} e^{b_{n,k}(1-\lambda_t)} (1-\lambda_t)^k + a_{n,n} (1-\lambda_t)^n \right]$$

Developing the recursions, we obtain:

$$\begin{aligned} a_{n,k} &= a_{n-1,k-1} \quad \text{for} \quad k > 0 \\ a_{n,0} &= a_{n-1,0} \cdot (1-\text{LGD}) \cdot e^{\Lambda_\delta - \kappa_\delta} \\ b_{n-k} &= -(n-k-1)\kappa_y \end{aligned} \tag{47}$$

Starting from $a_{0,0} = 1$, we have:

$$a_{n,k} = (1-\text{LGD})^{n-k} \cdot e^{(n-k)(\Lambda_\delta - \kappa_\delta)} \tag{48}$$

Thus we have:

$$\begin{aligned}
B_t^{(n)} &= e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)}\left[\lambda_t\sum_{k=0}^{n-1}(1-\mathrm{LGD})^{n-k}\cdot e^{(n-k)(\Lambda_\delta-\kappa_\delta)}e^{-(n-k-1)\kappa_y(1-\lambda_t)}\left(1-\lambda_t\right)^k+\left(1-\lambda_t\right)^n\right]\\
&= e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)}\left[(1-\mathrm{LGD})^n\cdot e^{n(\Lambda_\delta-\kappa_\delta-\kappa_y(1-\lambda_t))+\kappa_y(1-\lambda_t)}\lambda_t\sum_{k=0}^{n-1}\frac{\left(1-\lambda_t\right)^k}{\left((1-\mathrm{LGD})\,e^{\Lambda_\delta-\kappa_\delta-\kappa_y(1-\lambda_t)}\right)^k}+\left(1-\lambda_t\right)^n\right]\\
&= e^{n\left(\overline{M}-\kappa_0-\kappa_y\lambda_t\right)}\left[\left(1-\lambda_t\right)^n+e^{\kappa_y(1-\lambda_t)}\lambda_t\frac{(1-\mathrm{LGD})^n\cdot e^{n[\Lambda_\delta-\kappa_\delta-\kappa_y(1-\lambda_t)]}-\left(1-\lambda_t\right)^n}{1-\left(1-\lambda_t\right)(1-\mathrm{LGD})^{-1}e^{-[\Lambda_\delta-\kappa_\delta-\kappa_y(1-\lambda_t)]}}\right] \qquad (49)
\end{aligned}$$

Though complicated, the ILSBEI spread is available in closed-form and does not depend on $\overline{M}$.

We hereby derive the pricing formulas under the expectation hypothesis to obtain the risk premium. We start with the real riskfree bonds. We have:

$$\begin{aligned}
D_{t,EH}^{(1)^*} &= e^{-r_t^{(1)^*}} = e^{\overline{M}}\left[1+\lambda_t\left(e^{\Lambda_\delta}-1\right)\right]\\
D_{t,EH}^{(2)^*} &= e^{-r_t^{(1)^*}}\,\mathbb{E}_t\left[e^{-r_{t+1}^{(1)^*}}\right]\\
&= e^{-r_t^{(1)^*}}\cdot e^{\overline{M}}\cdot\mathbb{E}_t\left[1+\lambda_{t+1}\left(e^{\Lambda_\delta}-1\right)\right]\\
&= e^{-r_t^{(1)^*}}\cdot\left[\lambda_t\cdot e^{\overline{M}+\Lambda_\delta}+(1-\lambda_t)D_{t,EH}^{(1)^*}\right]
\end{aligned}$$

where the last row is obtained by separating what happens to the short-term rate in case of default and non-default. Following the recursion, we can easily show that:

$$D_{t,EH}^{(n)^*} = e^{-r_t^{(1)^*}}\left[\lambda_t e^{(n-1)\left(\overline{M}+\Lambda_\delta\right)}+(1-\lambda_t)D_{t,EH}^{(n-1)^*}\right]. \qquad (50)$$

In the same spirit, we can derive the recursions for the nominal riskless bonds:

$$\begin{aligned}
D_{t,EH}^{(1)} &= e^{-r_t^{(1)^*}}\,\mathbb{E}_t\left(e^{-\pi_{t+1}}\right) = e^{\overline{M}-\kappa_0-\kappa_y\lambda_t}\left[1+\lambda_t\left(e^{\Lambda_\delta}-1\right)\right]\left[1+\lambda_t\left(e^{-\kappa_\delta}-1\right)\right]\\
D_{t,EH}^{(2)} &= e^{-r_t^{(1)^*}}\,\mathbb{E}_t\left[e^{-\pi_{t+1}}D_{t+1,EH}^{(1)}\right]\\
&= e^{-r_t^{(1)^*}-\kappa_0-\kappa_y\lambda_t}\mathbb{E}_t\left[e^{-\kappa_\delta\mathbb{1}\left\{\delta_{t+1}^{(c)}>0\right\}-r_{t+1}^{(1)^*}-\kappa_0-\kappa_y\lambda_{t+1}}\left[1+\lambda_{t+1}\left(e^{-\kappa_\delta}-1\right)\right]\right]\\
&= e^{-r_t^{(1)^*}-\kappa_0-\kappa_y\lambda_t}\cdot\left[\lambda_t\cdot e^{\overline{M}+\Lambda_\delta-\kappa_0-\kappa_y-2\kappa_\delta}+(1-\lambda_t)D_{t,EH}^{(1)}\right]
\end{aligned}$$

Following the recursion again, we obtain:

$$D_{t,EH}^{(n)} = e^{-r_t^{(1)^*}-\kappa_0-\kappa_y\lambda_t}\left[\lambda_t e^{(n-1)\left(\overline{M}+\Lambda_\delta-\kappa_0-\kappa_y\right)-n\kappa_\delta}+(1-\lambda_t)D_{t,EH}^{(n-1)}\right]. \qquad (51)$$

Similarly, for defaultable bonds:

$$
\begin{aligned}
B_{t,EH}^{(1)*} &= e^{-r_t^{(1)*}} \left(\lambda_t \left[1-\mathrm{LGD}^*\right] + 1 - \lambda_t\right) = e^{\overline{M}} \left[1 + \lambda_t \left(e^{\Lambda_\delta} - 1\right)\right] \left[1 - \lambda_t \mathrm{LGD}^*\right] \\
B_{t,EH}^{(2)*} &= e^{-r_t^{(1)*}} \mathbb{E}_t \left[\left(\mathbb{1}\left\{\delta_{t+1}^{(c)} > 0\right\} (1-\mathrm{LGD}^*) + \mathbb{1}\left\{\delta_{t+1}^{(c)} = 0\right\} \right) B_{t+1,EH}^{(1)*} \right] \\
&= e^{-r_t^{(1)*}} \left[\lambda_t \left(1-\mathrm{LGD}^*\right) e^{\overline{M}+\Lambda_\delta} \left(1-\mathrm{LGD}^*\right) + (1-\lambda_t) B_{t,EH}^{(1)*} \right]
\end{aligned}
$$

Thus, we obtain:

$$
B_{t,EH}^{(n)*} = e^{-r_t^{(1)*}} \left[\lambda_t \left(1-\mathrm{LGD}^*\right) e^{(n-1)\left(\overline{M}+\Lambda_\delta\right)} + (1-\lambda_t) B_{t,EH}^{(n-1)*} \right]. \tag{52}
$$

and for nominal bonds:

$$
\begin{aligned}
B_{t,EH}^{(1)} &= e^{-r_t^{(1)*}} \mathbb{E}_t \left(e^{-\pi_{t+1}} \left(\mathbb{1}\left\{\delta_{t+1}^{(c)} > 0\right\} (1-\mathrm{LGD}) + \mathbb{1}\left\{\delta_{t+1}^{(c)} = 0\right\} \right) \right) \\
&= e^{-r_t^{(1)*} - \kappa_0 - \kappa_y \lambda_t} \left[\lambda_t (1-\mathrm{LGD}) e^{-\kappa_\delta} + 1 - \lambda_t \right] \\
&= e^{-r_t^{(1)*} - \kappa_0 - \kappa_y \lambda_t} \left[1 - \lambda_t \left(1 - (1-\mathrm{LGD}) e^{-\kappa_\delta} \right) \right] \\
B_{t,EH}^{(2)} &= e^{-r_t^{(1)*}} \mathbb{E}_t \left[e^{-\pi_{t+1}} \left(\mathbb{1}\left\{\delta_{t+1}^{(c)} > 0\right\} (1-\mathrm{LGD}) + \mathbb{1}\left\{\delta_{t+1}^{(c)} = 0\right\} \right) B_{t+1,EH}^{(1)} \right] \\
&= e^{-r_t^{(1)*} - \kappa_0 - \kappa_y \lambda_t} \left[\lambda_t (1-\mathrm{LGD}) e^{-\kappa_0 - \kappa_y + \overline{M} + \Lambda_\delta - 2\kappa_\delta} (1-\mathrm{LGD}) + (1-\lambda_t) B_{t,EH}^{(1)} \right]
\end{aligned}
$$

Thus:

$$
B_{t,EH}^{(n)} = e^{-r_t^{(1)*} - \kappa_0 - \kappa_y \lambda_t} \left[\lambda_t (1-\mathrm{LGD})^n e^{(n-1)\left(\overline{M}+\Lambda_\delta-\kappa_0-\kappa_y\right) - n\kappa_\delta} + (1-\lambda_t) B_{t,EH}^{(n-1)} \right] \tag{53}
$$

C Mechanism: Differential Loss Given Default

Let us focus on the mechanism involving a differential LGD and consider a differential treatment of nominal treasuries and TIPS in the case of default. Consider for instance a debt restructuring as in the case of the Greek 2012 default. The government proposed to bondholders to exchange outstanding treasuries against newly issued nominal bonds, regardless of potential inflation protection. The exchange was done face-value for face-value.

In this case, the loss given default is bond-specific. Imagine that the newly-issued bonds are valued 50cts per dollar of face value, then the holder of a bond trading at par obtains a recovery rate of exactly 50%, while the holder of a bond trading at a discount will obtain a higher recovery rate. TIPS tend to trade at a premium compared to equivalent-maturity nominals because of the inflation protection. However, their face value also increase during their lifetime with the accrued inflation accumulated from the past. Thus, what will eventually determine if TIPS suffer more than equivalent nominals in case of default depends on how the sovereign decides to treat the accrued past inflation in the face value of the bond. If the government forgives all inflation indexation, the TIPS will suffer more from default than its nominal counterpart since its face value reduces to one. This disindexation could explain the differential pricing of the defaultable treasuries by itself.

This channel has strong implications for the differential pricing of TIPS. Consider two TIPS that have been issued at different times, but have the same maturity date. Today's date is t, and the older TIPS has been issued at date 0 with semi-annual coupon c_{old}. The newer TIPS has just been issued and has semi-annual coupon c_{new}. Both bonds have remaining time to maturity T. Since all payment dates are aligned, a long-short position can easily annihilate all coupon cashflows. The individual positions are given by:

$$w_{old} = c_{new} \exp\left(-\sum_{i=1}^{t} \pi_i\right) \quad \text{and} \quad w_{new} = c_{old}\,, \tag{54}$$

and, assuming long position in the newer bond, the portfolio reduces to a zero coupon TIPS, where the final payment is given by:

$$\exp\left(\sum_{i=t+1}^{T} \pi_i\right)(c_{old} - c_{new})\ . \tag{55}$$

In other words the long-short portfolio reduces to a newly issued zero-coupon TIPS with maturity T and the face value is $(c_{old} - c_{new})$.

Let us assume that the sovereign defaults at date τ, and that the sovereign only honors a fraction ρ^* of the accrued inflation since inception of each bond. The recovery payments of the old and new TIPS are respectively given by:

$$w_{old} \exp\left(\rho^* \sum_{i=1}^{\tau} \pi_i\right) \quad \text{and} \quad w_{new} \exp\left(\rho^* \sum_{i=t+1}^{\tau} \pi_i\right)\ , \tag{56}$$

such that the recovery payment of the portfolio is given by:

$$\exp\left(\rho^* \sum_{i=t+1}^{\tau} \pi_i\right)\left[c_{old} - c_{new} \exp\left((\rho^* - 1)\sum_{i=1}^{t} \pi_i\right)\right]. \tag{57}$$

As a last step, we add a zero-coupon TIPS position to the portfolio, with face value $(c_{new} - c_{old})$ to cancel the principal repayment. Assuming homogeneity of treatment for all TIPS bonds in case of default, the recovery payment of the aggregated portfolio is given by:

$$RP = \exp\left(\rho^* \sum_{i=t+1}^{\tau} \pi_i\right) c_{new}\left[1 - \exp\left((\rho^* - 1)\sum_{i=1}^{t} \pi_i\right)\right]. \tag{58}$$

Provided accrued inflation over long periods is usually positive, we expect the recovery payment to be positive since the term in the squared brackets is positive. Notice that the recovery payment grows with the degree of disindexation $(1 - \rho^*)$ and goes to zero when $\rho^* = 1$. This theory can thus be tested by forming the price of these portfolios and tracking their prices over time.

We identify 5 TIPS pairs with aligned maturity dates that have been issued at different dates. We summarize their characteristics in Table A8. Note that these bonds all pay semi-annually and have the same seniority level. For each bond pairs, we compute the weights of the long-short portfolio using their coupon rate and a reconstructed series of the CPI-U reference index taking into account its indexation lag. The complete portfolio necessitates the price of a defaultable zero-coupon TIPS for any possible time to maturity in days. We use the smoothed Nelson-Siegel-Svensson parameters provided by Gurkaynak, Sack and Wright (2010) and reconstruct the zero-coupon inflated price for each date.

The results are presented on Figure A1. All 5 bond pairs show the same pattern, and the mispricing is high when the new bond is issued due to the on-the-run premium, but decreases rapidly afterwards. All bond pairs eventually fluctuate mostly in the ±25cts range for \$100 face value. Bond pair 5, the most recent one in our sample, is a bit more elevated and fluctuates between 25cts to 50cts for \$100 face value. These values are overall small. To check that they could be the result of the smoothed zero-coupon model, we compute confidence bands by adding or subtracting 3bps to the TIPS zero-coupon bond yield. All confidence bands include 0. This casts doubt on this particular channel as being the main driver of our results.

D A Lucas tree economy with endogenous inflation

D.1 Model

Let us consider a CRRA Lucas tree economy with the following dynamics:

$$\begin{aligned} \Delta c_{t+1} &= x_t + \sigma_c \varepsilon_{t+1} + \kappa_c \delta_{t+1}^{(c)} \\ x_{t+1} &= \rho_x x_t + \varphi_\delta \delta_t^{(c)} + \varphi_\lambda \lambda_t + \sigma_x \varepsilon_{t+1} \end{aligned} \tag{59}$$

where there is no intercept in the consumption dynamics without loss of generality, and ε_t is standard normal. The default dynamics are given by:

$$\begin{aligned} \mathbb{P}_t\left(\delta_{t+1}^{(c)} > 0\right) &= \lambda_t \\ \lambda_{t+1} &= \overline{\lambda} + \rho_\lambda \lambda_t + \sigma_\lambda \epsilon_{t+1} \end{aligned} \tag{60}$$

where ϵ_t is standard normal, and $\delta_t^{(c)}$ is gamma-zero distributed with scale parameter 1.[8] In addition, we assume that the monetary policy follows a standard Taylor rule, such that:

$$i_t = \bar{i} + b_\pi \pi_t + b_x x_t \, . \tag{61}$$

The pricing kernel is given by:

$$M_{t+1} = \beta e^{-\gamma \Delta c_{t+1}} \, . \tag{62}$$

We assume that the inflation rate has a linear formulation given the states, such that:

$$\pi_t = \overline{\pi} + \kappa_x x_t + \kappa_\lambda \lambda_t + \kappa_\delta \delta_t^{(c)} \, . \tag{63}$$

We solve these coefficients as a function of the remaining dynamics.

[8]We could alternatively design the λ_t process as an autoregressive gamma and make the default probability $1 - e^{-\lambda_t}$, as is the case for the term structure model. This merely complicates computations.

D.2 Solution

We calculate the nominal riskless bond price as:

$$
\begin{aligned}
\mathbb{E}_t\left(M_{t+1}e^{-\pi_{t+1}}\right) &= \beta\mathbb{E}_t\left[\exp\left\{-\gamma x_t - \gamma\sigma_c\varepsilon_{t+1} - \gamma\kappa_c\delta_{t+1}^{(c)} - \overline{\pi} - \kappa_x\left(\rho_x x_t + \varphi_\delta\delta_t^{(c)} + \varphi_\lambda\lambda_t + \sigma_x\varepsilon_{t+1}\right)\right.\right. \\
&\qquad \left.\left. -\kappa_\lambda\lambda_{t+1} - \kappa_\delta\delta_{t+1}^{(c)}\right\}\right] \\
&= \beta\exp\left\{-\overline{\pi} + \frac{(\kappa_x\sigma_x + \gamma\sigma_c)^2}{2} - (\gamma + \kappa_x\rho_x)\,x_t - \kappa_x\varphi_\delta\delta_t^{(c)} - \kappa_x\varphi_\lambda\lambda_t\right\}\times \\
&\qquad \mathbb{E}_t\left[\exp\left\{-\left(\gamma\kappa_c + \kappa_\delta\right)\delta_{t+1}^{(c)} - \kappa_\lambda\lambda_{t+1}\right\}\right] \\
&= \beta\exp\left\{-\overline{\pi} + \frac{(\kappa_x\sigma_x + \gamma\sigma_c)^2}{2} - (\gamma + \kappa_x\rho_x)\,x_t - \kappa_x\varphi_\delta\delta_t^{(c)} - \kappa_x\varphi_\lambda\lambda_t\right\}\times \\
&\qquad \exp\left\{-\frac{\gamma\kappa_c + \kappa_\delta}{1 + \gamma\kappa_c + \kappa_\delta}\lambda_t - \kappa_\lambda\left(\overline{\lambda} + \rho_\lambda\lambda_t\right) + \frac{\kappa_\lambda^2\sigma_\lambda^2}{2}\right\}
\end{aligned}
$$

Therefore we have:

$$
\begin{aligned}
i_t &= -\log\beta + \overline{\pi} + \kappa_\lambda\overline{\lambda} - \frac{(\kappa_x\sigma_x + \gamma\sigma_c)^2}{2} - \frac{\kappa_\lambda^2\sigma_\lambda^2}{2} \\
&\quad + (\gamma + \kappa_x\rho_x)\,x_t + \kappa_x\varphi_\delta\delta_t^{(c)} + \left(\kappa_x\varphi_\lambda + \frac{\gamma\kappa_c + \kappa_\delta}{1 + \gamma\kappa_c + \kappa_\delta} + \kappa_\lambda\rho_\lambda\right)\lambda_t
\end{aligned} \tag{64}
$$

The nominal interest rate is also determined by the Taylor rule, such that:

$$
i_t = \overline{i} + b_\pi\left(\overline{\pi} + \kappa_x x_t + \kappa_\lambda\lambda_t + \kappa_\delta\delta_t^{(c)}\right) + b_x x_t\,. \tag{65}
$$

Putting both equations together, we have the following system:

$$
\overline{i} + b_\pi\overline{\pi} = -\log\beta + \overline{\pi} + \kappa_\lambda\overline{\lambda} - \frac{(\kappa_x\sigma_x + \gamma\sigma_c)^2}{2} - \frac{\kappa_\lambda^2\sigma_\lambda^2}{2} \tag{66}
$$

$$
b_x + b_\pi\kappa_x = \gamma + \kappa_x\rho_x \tag{67}
$$

$$
b_\pi\kappa_\delta = \kappa_x\varphi_\delta \tag{68}
$$

$$
b_\pi\kappa_\lambda = \kappa_x\varphi_\lambda + \kappa_\lambda\rho_\lambda + \frac{\gamma\kappa_c + \kappa_\delta}{1 + \gamma\kappa_c + \kappa_\delta} \tag{69}
$$

As long as $b_\pi \neq 1$, $\overline{\pi}$ is pinned down by Equation (66). We then obtain:

$$\kappa_x = \frac{\gamma - b_x}{b_\pi - \rho_x}\,, \quad \kappa_\delta = \frac{\varphi_\delta}{b_\pi} \cdot \kappa_x = \frac{\varphi_\delta}{b_\pi} \cdot \frac{\gamma - b_x}{b_\pi - \rho_x}$$

$$\text{and} \quad \kappa_\lambda = \frac{1}{b_\pi - \rho_\lambda}\left[\kappa_x \varphi_\lambda + \frac{\gamma\kappa_c + \kappa_\delta}{1 + \gamma\kappa_c + \kappa_\delta}\right]. \tag{70}$$

D.3 Discussion

Assuming the system to be stationary such that $|\rho_x| < 1$ and $|\rho_\lambda| < 1$, the inflation jump upon default parameter κ_δ is:

$$\kappa_\delta = \frac{\varphi_\delta}{b_\pi} \cdot \kappa_x\,. \tag{71}$$

b_π is naturally positive since it represents the strength of the central ban's battle against inflation. Its magnitude is key to determining the inflation parameters in the system. Economic intuition also suggests that φ_δ is negative, implying that the long-run mean of consumption growth expectation decreases persistently after a sovereign default occurs. The magnitude of φ_δ provides the size of the initial forecast decrease, which is smoothed for the subsequent periods by the parameter ρ_x. Therefore, the sign of κ_δ is that of $-\kappa_x$, which is examined below.

The impact of long run consumption growth on inflation is given by:

$$\kappa_x = \frac{\gamma - b_x}{b_\pi - \rho_x}\,. \tag{72}$$

In standard setups, the price of risk γ will typically be larger than the central bank reaction coefficient for real activity. Common values for the latter are 0.5 or 1. Therefore, the sign of κ_x is the same as that of $b_\pi - \rho_x$. Since the long run component of consumption growth is fairly persistent ($\rho_x = 0.97$ in the original Bansal & Yaron 2004 calibration), the size of the inflation coefficient in the Taylor rule, b_π, is the key determinant of inflation reaction to consumption growth, and default.

If $b_\pi < \rho_x$, then $\kappa_x < 0$, leading to hyperinflation upon default since $\kappa_\delta > 0$. Notice that the direct and indirect effects of default on inflation add to each other. Indeed, when $\delta_t^{(c)}$ jumps, inflation increases by κ_δ. On the next period, x_{t+1} decreases by φ_δ, meaning that inflation increases further at $t+1$ since κ_x is negative. Both effects thus play in the same direction.

The relationship between the default probability and inflation is governed by the coefficient κ_λ, which has the following function form:

$$\kappa_\lambda = \frac{1}{b_\pi - \rho_\lambda}\left[\kappa_x \varphi_\lambda + \frac{\gamma\kappa_c + \kappa_\delta}{1 + \gamma\kappa_c + \kappa_\delta}\right]. \tag{73}$$

To simplify the discussion, we assume that $\varphi_\lambda = 0$. Since the default probability is very persistent, we can safely assume that ρ_λ will be close to 1 such that $b_\pi - \rho_\lambda$ is negative. Thus, Equation (73) shows that κ_λ has the opposition sign as the fraction involving the impact of default on inflation (κ_δ) and on consumption growth (κ_c). This ratio is positive as long as:

$$\kappa_\delta > -\gamma\kappa_c\,. \tag{74}$$

The previous condition implies that the inflation jump upon default should be sufficiently large.[9] This condition will lead to a negative correlation between inflation and the default probability. Notably, this means that κ_δ and κ_λ can be of different signs, and inflation can react in opposite ways to default probability and the default event itself.

As long-run consumption growth expectation (x_t) falls, inflation will react according to κ_x. If κ_x is positive, this implies that inflation is also falling, and the central bank will adjust the nominal interest rate down according to the Taylor rule. Because the consumption growth process is persistent, low expected growth today translates into low expected growth tomorrow. Low expected consumption growth tomorrow means high marginal utility state of the world causing the real interest rate to drop. In this scenario, both real and nominal rates decrease as consumption growth expectations decline. On the other hand, if κ_x is negative, inflation and the nominal interest rate both rise when x_t drops. Increase in expected inflation causes the nominal stochastic discount factor to decrease. This makes nominal bonds cheaper and nominal rates higher. In order for the Euler equation for nominal bonds to continue to hold, it has to be the case that the decline in expected consumption growth, after scaling by γ, cannot dominate the rise in expected inflation. Otherwise, nominal bond prices rise and push down the nominal interest rate. Futhermore, the solution also requires that the central bank does not respond to the rise in inflation too strongly, i.e. less than one-for-one or $b_\pi < 1$, since the falling real interest rate makes it impossible for the nominal rate to rise faster than inflation. In this world, an increase in expected consumption growth will make real interest rates increase, which will lead inflation downwards. In reaction, the central bank will decrease its nominal interest rate to mitigate the effects on inflation.

[9] Note that to be well-defined, the Laplace transform needs that $\gamma\kappa_c + \kappa_\delta > -1$.

When a sovereign default happens, the current expected consumption growth (x_t) stays the same but its future path is revised downwards because of $\varphi_\delta < 0$. Accordingly, the real interest rate is not going to move but expected inflation will increase, such that the equilibrium nominal rate will increase as well. Therefore, current inflation adjusts upwards for the Taylor rule nominal rate to go up as well.

Last, an increase in the default probability makes expected consumption growth decline through κ_c, as the default process directly enters the realized consumption growth process. The equilibrium real rate will thus adjust downwards. As the chance of a default is higher, inflation has a higher probability to jump by κ_δ so expected inflation is revised upwards. The effect on the equilibrium nominal rate can thus be positive or negative. If the inflation jump upon default is very large, the equilibrium nominal rate will increase as well as the Taylor rule policy rate. However, when the reaction coefficient to inflation is low enough (b_π), the Taylor rule interest rate can never catch up with the equilibrium nominal rate so inflation needs to go down in reaction. This generates the negative correlation between inflation and default probability.

E Closed-Form Estimation Equations

E.1 The term structure of ILS

Our first empirical target is the zero-coupon inflation-linked swaps. We assume in our model that ILS are virtually riskfree, such that they are equivalent to a long-short position on nominal and real bonds. By no-arbitrage, the swap rate of a n residual maturity is given by:

$$\mathrm{ILS}_t(n) = \frac{1}{n} \log \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(\sum_{i=1}^{n} \pi_{t+i} \right) \right] . \tag{75}$$

Using the inflation specification (10) and the risk-neutral dynamics of the factors, we easily find that:

$$\mathrm{ILS}_t(n) = \mathsf{a}_{ils,n} + \mathsf{b}_{ils,n}^{(x)\prime} \cdot x_t + \mathsf{b}_{ils,n}^{(y)\prime} \cdot y_t , \tag{76}$$

where the loadings are obtained through closed-form recursions (See Appendix H). If inflation does not depend on y_t or δ_t, i.e. when $\kappa_y^{(\pi)} = 0$ and $\kappa_\delta^{(\pi)} = 0$, the last term of Equation (76) disappears and all the riskless yield curves are explained by x_t. If in turn inflation depends on the default and credit-event variables, then $\mathsf{b}_{ils,n}^{(y)}$ is different from zero and credit factors and the default probability enter the ILS curve. This feature results from the inflation and the pricing kernel specification. Indeed, both riskless nominal and real bonds do not suffer from sovereign default directly, and are hence hedges against default risk. However, nominals can be more exposed to default risk than real bonds through inflation risk. If inflation correlates positively with default risk, nominal bonds are more exposed to default than real bonds ($\mathsf{b}^{(y)}_{ils,n} > 0$), since their real cashflow decreases significantly in case of default, and vice versa. Last, notice that since Φ_y is lower triangular and inflation does not depend on the liquidity factor $y^{(\ell)}_t$, this factor does not enter the ILS curve.

Our empirical exercise also considers OIS rates as proxies for riskless nominal yields. It is easy to show that the yield of a riskless nominal bond of residual maturity n, denoted by $r_t(n)$, is given by:

$$r_t(n) = \mathsf{a}_{rf,n} + \mathsf{b}^{(x)'}_{rf,n} \cdot x_t + \mathsf{b}^{(y)'}_{rf,n} \cdot y_t \, , \tag{77}$$

where the loadings are also obtained through closed-form recursions (See Appendix H).

E.2 Treasuries and default events

The term structures of nominal treasuries and TIPS constitutes our second empirical target. By no-arbitrage, the prices of these securities can be obtained by computing the present value of all future cashflows. However, we need to include the possibility that the sovereign may default and not entirely repay the principals associated with the treasuries investments.

Having defined the cashflows in all states of the world (see Section 4.4 and the summary of Table A9), we can provide the general pricing formulas. The price of a TIPS of residual maturity n is given by:

$$\begin{aligned} B^*_t(n) = \quad & \sum_{i=1}^{n} \quad \mathbb{E}^{\mathbb{Q}}_t \left[\exp\left(-\delta^{(c)}_{t+i} - \sum_{j=0}^{i-1} \left[r^{(1)^*}_{t+j} + (1-\rho^*)\pi_{t+j+1} \right] \right) \cdot \mathbb{1}\left\{ \sum_{j=0}^{i-1} \delta^{(c)}_{t+j} + \delta^{(\ell)}_{t+j} = 0 \right\} \mathbb{1}\left\{ \delta^{(c)}_{t+i} > 0 \right\} \right] \\ & + \quad \mathbb{E}^{\mathbb{Q}}_t \left[\exp\left(-\delta^{(\ell)}_{t+i} - \sum_{j=0}^{i-1} r^{(1)^*}_{t+j} \right) \cdot \mathbb{1}\left\{ \sum_{j=0}^{i-1} \delta^{(c)}_{t+j} + \delta^{(\ell)}_{t+j} = 0 \right\} \mathbb{1}\left\{ \delta^{(c)}_{t+i} = 0 \cap \delta^{(\ell)}_{t+i} > 0 \right\} \right] \\ & + \quad \mathbb{E}^{\mathbb{Q}}_t \left[\exp\left(- \sum_{j=0}^{n-1} r^{(1)^*}_{t+j} \right) \mathbb{1}\left\{ \sum_{j=0}^{n} \delta^{(c)}_{t+j} + \delta^{(\ell)}_{t+j} = 0 \right\} \right] \, . \end{aligned} \tag{78}$$

The first row of (78) represents the discounted cashflow in case of a credit event at $t+i$, the second row has the similar interpretation for a liquidity event, and the last row gives the present value of the consumption unit at maturity.[10] In the same spirit, the price of a nominal treasury of residual maturity n is given by:

$$\begin{aligned} B_t(n) = & \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)} - \sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*} + \pi_{t+j+1}\right)\right) \cdot \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)} = 0\right\} \mathbb{1}\left\{\delta_{t+i}^{(c)} > 0\right\}\right] \\ + & \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1}\left(r_{t+j}^{(1)^*} + \pi_{t+j+1}\right)\right) \mathbb{1}\left\{\sum_{j=0}^{n}\delta_{t+j}^{(c)} = 0\right\}\right] . \end{aligned} \tag{79}$$

Equation (79) simply states that the price of the nominal bond is the sum of discounted recovery payments if default happens between $t+i-1$ and $t+i$ (first row), and the discounted principal if no default occurs during the lifespan of the bond (second row).

We show in Appendix I that our model provides closed-form (though, non-affine) pricing formulas for Equations (79-78), such that the prices of both bonds, are given by:

$$\begin{aligned} B_t^*(n) &= \sum_{i=1}^{n} \Big[\exp\left(\widetilde{\mathsf{A}}_{tips,i} + \widetilde{\mathsf{B}}_{tips,i}^{(x)'} \cdot x_t + \widetilde{\mathsf{B}}_{tips,i}^{(y)'} \cdot y_t\right) - \exp\left(\mathsf{A}_{tips,i} + \mathsf{B}_{tips,i}^{(x)'} \cdot x_t + \mathsf{B}_{tips,i}^{(y)'} \cdot y_t\right) \\ &+ \exp\left(\widetilde{\mathsf{C}}_{tips,i} + \widetilde{\mathsf{D}}_{tips,i}^{(x)'} \cdot x_t + \widetilde{\mathsf{D}}_{tips,i}^{(y)'} \cdot y_t\right) - \exp\left(\mathsf{C}_{tips,i} + \mathsf{D}_{tips,i}^{(x)'} \cdot x_t + \mathsf{D}_{tips,i}^{(y)'} \cdot y_t\right)\Big] \\ &+ \exp\left(\mathsf{C}_{tips,n} + \mathsf{D}_{tips,n}^{(x)'} \cdot x_t + \mathsf{D}_{tips,n}^{(y)'} \cdot y_t\right) , \end{aligned} \tag{80}$$

and:

$$\begin{aligned} B_t(n) &= \sum_{i=1}^{n} \exp\left(\widetilde{\mathsf{A}}_{nom,i} + \widetilde{\mathsf{B}}_{nom,i}^{(x)'} \cdot x_t + \widetilde{\mathsf{B}}_{nom,i}^{(y)'} \cdot y_t\right) - \exp\left(\mathsf{A}_{nom,i} + \mathsf{B}_{nom,i}^{(x)'} \cdot x_t + \mathsf{B}_{nom,i}^{(y)'} \cdot y_t\right) \\ &+ \exp\left(\mathsf{A}_{nom,n} + \mathsf{B}_{nom,n}^{(x)'} \cdot x_t + \mathsf{B}_{nom,n}^{(y)'} \cdot y_t\right) . \end{aligned} \tag{81}$$

We can then easily obtain the BEI pricing formula by considering the log-difference of TIPS and

[10] We leave aside the embedded option and the inflation lag for simplicity in these pricing formulas. First, note that the embedded inflation option would, if anything, raise the price of the TIPS, decrease its yield, thus play against a large ILSBEI spread. Hence, by neglecting the deflation option, we underestimate the role of the other factors, if anything. In addition, notice that the value of this deflation floor would be the biggest during the financial crisis, where the ILSBEI spread is the biggest. Our simplification is thus conservative. Second, while the inflation lag can matter for short-enough maturities (below 2y), it is unlikely to have a large impact for longer maturities since the 3-months lag represents a smaller proportion of the total maturity of the bond.

nominal bond prices:

$$\text{BEI}_t(n) \quad = \quad \frac{1}{n}\log\left(\frac{B_t^*(n)}{B_t(n)}\right). \tag{82}$$

E.3 CDS pricing

In case of default, the payment provided by the CDS is equal to the loss given default of a nominal bond, i.e. $1-e^{-\delta_t^{(c)}}$. The present value of the protection is given by:

$$\text{PS}_t(n)=\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*}+\pi_{t+j+1}\right)\right)\left(1-e^{-\delta_{t+i}^{(c)}}\right)\mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}=0\right\}\mathbb{1}\left\{\delta_{t+i}^{(c)}>0\right\}\right].$$

We assume that a buyer of protection makes periodic payments from time t to maturity n to protect against any type of credit event. The cash flow payment at time $t+i$ conditional on no default is designated as $\mathcal{S}_t^{(n)}$. The present value of the stream of cash flows paid by the protection buyer is:

$$\text{PB}_t^{(n)}=\mathcal{S}_t^{(n)}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*}+\pi_{t+j+1}\right)\right)\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}=0\right\}\right]$$

No arbitrage pricing requires that the present value of the protection bought is equal to the present value of the protection sold. Equating both legs at inception, the swap spread yields:

$$\mathcal{S}_t^{(n)}=\frac{\widetilde{B_t}(n)-B_t(n)}{\sum_{i=1}^{n}\exp\left(\mathsf{A}_{nom,i}+\mathsf{B}_{nom,i}^{(x)'}\cdot x_t+\mathsf{B}_{nom,i}^{(y)'}\cdot y_t\right)} \tag{83}$$

where $\widetilde{B_t}(n)$ represents the price of a nominal bond for a recovery rate of 100%, and $B_t(n)$ is the exact pricing formula presented in Equation (81) (see Appendix J).

F Affine property and conditional moments of w_t

Let us compute the physical conditional moment-generating function of $w_t=(x_t',y_t',\delta_t')'$ applied in $u=\left(u_x',\,u_y',\,u_\delta'\right)'$. For the sake of generality we provide the formulas for $\varepsilon_{x,t}\sim\mathcal{N}(0,\Sigma)$ and general scale parameter c_y. In practice our empirical model assumes $\Sigma = I_3$.

$$
\begin{aligned}
\varphi^{\mathbb{P}}_{w_t}(u) &:= \mathbb{E}^{\mathbb{P}}_t\left[\exp\left(u' w_{t+1}\right)\right] \\
&= \exp\left\{u_x'\left(\mu + \Phi_x x_t\right) + \frac{1}{2}u_x'\Sigma u_x\right\}\mathbb{E}^{\mathbb{P}}_t\left\{\mathbb{E}^{\mathbb{P}}_t\left[\exp\left(u_y' y_{t+1} + u_\delta'\delta_{t+1}\right)\middle| y_{t+1}\right]\right\} \\
&= \exp\left\{u_x'\left(\mu + \Phi_x x_t\right) + \frac{1}{2}u_x'\Sigma u_x\right\}\mathbb{E}^{\mathbb{P}}_t\left[\exp\left(\left(\beta_\lambda \frac{\mathrm{diag}(\mathsf{c}_\delta)u_\delta}{\mathbf{1} - \mathrm{diag}(\mathsf{c}_\delta)u_\delta} + u_y\right)' y_{t+1}\right)\right]
\end{aligned}
$$

where the fraction is an abuse of notation for an element by element ratio and:

$$
\beta_\lambda = \begin{pmatrix} \beta^{(c)}_\lambda & \mathbf{0} \\ \mathbf{0} & \beta^{(\ell)}_\lambda \end{pmatrix} .
$$

Thus, denoting by $\widetilde{u}_y = \left(\beta_\lambda \frac{\mathrm{diag}(\mathsf{c}_\delta)u_\delta}{\mathbf{1}-\mathrm{diag}(\mathsf{c}_\delta)u_\delta} + u_y\right)$, we have:

$$
\varphi^{\mathbb{P}}_{w_t}(u) = \exp\left\{u_x'\left(\mu + \Phi_x x_t\right) + \frac{1}{2}u_x'\Sigma u_x + \left(\frac{\mathrm{diag}(\mathsf{c}_y)\widetilde{u}_y}{\mathbf{1} - \mathrm{diag}(\mathsf{c}_y)\widetilde{u}_y}\right)' \Phi_y\, y_t - \nu' \log\left[\mathbf{1} - \mathrm{diag}(\mathsf{c}_y)\widetilde{u}_y\right]\right\} , \tag{84}
$$

which is an exponential-affine function of x_t and y_t, thus of w_t by extension. The conditional mean of w_t is then given by:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}_t\left(x_{t+1}\right) &= \mu + \Phi_x\, x_t \\
\mathbb{E}^{\mathbb{P}}_t\left(y_{t+1}\right) &= \mathrm{diag}(\mathsf{c}_y)\left(\nu + \Phi_y\, y_t\right) \\
\mathbb{E}^{\mathbb{P}}_t\left(\delta_{t+1}\right) &= \mathbb{E}^{\mathbb{P}}_t\left[\mathbb{E}^{\mathbb{P}}_t\left(\delta_{t+1}\middle| y_{t+1}\right)\right] = \mathrm{diag}(\mathsf{c}_\delta)\beta_\lambda'\mathbb{E}^{\mathbb{P}}_t\left(y_{t+1}\right) .
\end{aligned}
$$

For notational convenience, we introduce the block matrix Q of size $N \times N$ defined as:

$$
Q = \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}\left(\mathsf{c}_y\right) & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}\left(\mathsf{c}_\delta\right)\,\beta_\lambda\,\mathrm{diag}\left(\mathsf{c}_y\right) & \mathrm{diag}\left(\mathsf{c}_\delta\right) \end{pmatrix}
$$

We obtain that

$$
\mathbb{E}^{\mathbb{P}}_t(w_{t+1}) = \Psi_0 + \Psi\, w_t ,
$$

where:

$$\Psi_0 = Q \times \begin{pmatrix} \mu \\ \nu \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad \Psi = Q \times \begin{pmatrix} \Phi_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Phi_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} . \tag{85}$$

Let us now turn to the conditional variance. x_t is independent from y_t and δ_t, and its conditional covariance matrix is given by Σ. Then, using the properties of gamma variables, we have:

$$\mathbb{V}_t^{\mathbb{P}}\left(y_{t+1}\right) = \text{diag}\left(\mathsf{c}_y\right)^2 \times \text{diag}\left(\nu + 2\Phi_y\, y_t\right) .$$

Using the law of total variance, we can express the conditional variance of δ_t as:

$$\begin{aligned}
\mathbb{V}_t^{\mathbb{P}}\left(\delta_{t+1}\right) &= \mathbb{V}_t^{\mathbb{P}}\left[\mathbb{E}_t^{\mathbb{P}}\left(\delta_{t+1}\middle|y_{t+1}\right)\right] + \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{V}_t^{\mathbb{P}}\left(\delta_{t+1}\middle|y_{t+1}\right)\right] \\
&= \mathbb{V}_t^{\mathbb{P}}\left[\text{diag}\left(\mathsf{c}_\delta\right)\beta_\lambda'\, y_{t+1}\right] + \mathbb{E}_t^{\mathbb{P}}\left[2\text{diag}\left(\mathsf{c}_\delta\right)^2 \text{diag}\left(\beta_\lambda'\, y_{t+1}\right)\right] \\
&= \text{diag}\left(\mathsf{c}_\delta\right)\beta_\lambda'\, \mathbb{V}_t^{\mathbb{P}}(y_{t+1})\, \beta_\lambda \text{diag}\left(\mathsf{c}_\delta\right) + 2\text{diag}\left(\mathsf{c}_\delta\right)^2 \text{diag}\left(\beta_\lambda'\, \mathbb{E}_t^{\mathbb{P}}\left[y_{t+1}\right]\right) \\
&= \text{diag}\left(\mathsf{c}_\delta\right)\beta_\lambda'\, \mathbb{V}_t^{\mathbb{P}}(y_{t+1})\, \beta_\lambda \text{diag}\left(\mathsf{c}_\delta\right) + 2\text{diag}\left(\mathsf{c}_\delta\right)^2 \text{diag}\left[\beta_\lambda'\, \text{diag}\left(\mathsf{c}_y\right)\left(\nu + \Phi_y\, y_t\right)\right] .
\end{aligned}$$

Last, the conditional covariance between y_t and δ_t is given by:

$$\begin{aligned}
\mathbb{C}ov_t^{\mathbb{P}}\left(y_{t+1},\, \delta_{t+1}\right) &= \mathbb{C}ov_t^{\mathbb{P}}\left(y_{t+1},\, \mathbb{E}_t^{\mathbb{P}}\left[\delta_{t+1}\middle|y_{t+1}\right]\right) + \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{C}ov_t^{\mathbb{P}}\left(y_{t+1},\, \delta_{t+1}\middle|y_{t+1}\right)\right] \\
&= \mathbb{C}ov_t^{\mathbb{P}}\left(y_{t+1},\, \text{diag}\left(\mathsf{c}_\delta\right)\beta_\lambda'\, y_{t+1}\right) \\
&= \mathbb{V}_t^{\mathbb{P}}\left(y_{t+1}\right)\beta_\lambda\, \text{diag}\left(\mathsf{c}_\delta\right) .
\end{aligned}$$

Putting all results together, we obtain:

$$\Omega_{t-1} = \mathbb{V}_t^{\mathbb{P}}\left(w_{t+1}\right) = Q \times \begin{pmatrix} \Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{diag}\left(\nu + 2\Phi_y\, y_t\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\text{diag}\left[\beta_\lambda'\, \text{diag}\left(\mathsf{c}_y\right)\left(\nu + \Phi_y\, y_t\right)\right] \end{pmatrix} \times Q' . \tag{86}$$

We obtain unconditional moments by assuming stationarity of w_t:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left(w_t\right) &= \left(I_N - \Psi\right)^{-1}\Psi_0 \\
\text{Vec}\left[\mathbb{V}^{\mathbb{P}}\left(w_t\right)\right] &= \left[I_{N^2} - \left(Q \otimes Q\right)\left(\Psi \otimes \Psi\right)\right]^{-1} \times \left[\Omega_0 + \Omega\mathbb{E}^{\mathbb{P}}\left(y_t\right)\right] ,
\end{aligned}$$

where Ω_0 and Ω are such that:

$$\text{Vec}\begin{pmatrix} \Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{diag}\,(\nu + 2\Phi_y\, y_t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\text{diag}\,[\beta_\lambda'\, \text{diag}\,(\mathsf{c}_y)\,(\nu + \Phi_y\, y_t)] \end{pmatrix} = \Omega_0 + \Omega y_t\,.$$

G Affine risk-neutral property

To obtain the affine property under $\mathbb{Q}$ we need to proceed to the change of measure implied by the SDF specification of Equation (9). Since x_t is independent from y_t and δ_t and that the SDF does not incorporate cross-terms, we can proceed to its change of measure separately from that of $(y_t,\, \delta_t)$.

Our specification of x_t dynamics and the SDF that depends on x_t is that of a standard Gaussian affine term structure model with time varying prices of risk. We can thus directly apply the standard result that:

$$x_t = \mu^{\mathbb{Q}} + \Phi_x^{\mathbb{Q}} x_{t-1} + \sqrt{\Sigma}\,\varepsilon_t^{\mathbb{Q}}\,, \quad \text{where} \quad \varepsilon_t^{\mathbb{Q}} \sim \mathcal{N}\left(0, I_3\right)\,, \tag{87}$$

and the risk-neutral parameters are given by:

$$\mu^{\mathbb{Q}} = \mu + \Sigma\,\Lambda_{0,x}, \quad \Phi_x^{\mathbb{Q}} = \Phi_x + \Sigma\,\Lambda_{1,x}\,. \tag{88}$$

For the change of measure associated with the default and liquidity risk variables, we rely on Propositions 2.5-2.6 of Monfort et al. (2020), and we have that the risk-neutral intensities are proportional to the physical intensities:

$$\delta_t^{(c)} = \sum_{j=1}^{\mathsf{P}_t^{(c)}} \xi_{j,t}^{(c)} \quad \text{where} \quad \mathsf{P}_t^{(c)}\,|\,\lambda_t^{(c)} \overset{\mathbb{Q}}{\sim} \mathcal{P}\left(\lambda_t^{(c)^{\mathbb{Q}}}\right) \quad \text{and} \quad \xi_{j,t}^{(c)} \overset{\mathbb{Q}}{\sim} \mathcal{E}xp\left(\frac{1}{\mathsf{c}_\delta^{(c)^{\mathbb{Q}}}}\right)\,, \tag{89}$$

where

$$\lambda_t^{(c)^{\mathbb{Q}}} = \beta_{\lambda,1}^{(c)^{\mathbb{Q}}} \cdot y_{1,t}^{(c)} + \beta_{\lambda,2}^{(c)^{\mathbb{Q}}} \cdot y_{2,t}^{(c)} \tag{90}$$

and,

$$\beta_\lambda^{(c)^{\mathbb{Q}}} = \beta_\lambda^{(c)}\left(1 + c_\delta^{(c)^{\mathbb{Q}}}\Lambda_\delta\right), \quad \text{and} \quad c_\delta^{(c)^{\mathbb{Q}}} = \frac{c_\delta^{(c)}}{1 - c_\delta^{(c)}\Lambda_\delta} \tag{91}$$

and the risk-neutral intensity of the liquidity event variable is the same under both measures since $\delta_t^{(\ell)}$ is not in the SDF. The risk-neutral dynamics of y_t is given by a vector autoregressive gamma process, such that:

$$y_t \,|\, y_{t-1} \overset{\mathbb{Q}}{\sim} \Gamma_\nu \left(\Phi_y^{\mathbb{Q}} \, y_{t-1}, \, \mathsf{c}_y^{\mathbb{Q}} \right) \,, \tag{92}$$

where

$$\Phi_y^{\mathbb{Q}} = \operatorname{diag}\left(\frac{\mathsf{c}_y^{\mathbb{Q}}}{\mathsf{c}_y} \right) \cdot \Phi_y \quad \text{and} \quad \mathsf{c}_y^{\mathbb{Q}} = \frac{\mathsf{c}_y}{\mathbf{1} - \operatorname{diag}\left(\mathsf{c}_y\right) \left[\Lambda_y + \beta_\lambda \left(\frac{\operatorname{diag}(\mathsf{c}_\delta)\tilde{\Lambda}_\delta}{\mathbf{1} - \operatorname{diag}(\mathsf{c}_\delta)\tilde{\Lambda}_\delta} \right) \right]} \,, \tag{93}$$

where $\tilde{\Lambda}_\delta = (\Lambda_\delta, \, 0)'$. The proof is easily obtained below:

$$\mathbb{E}_{t-1}^{\mathbb{Q}} \left[\exp\left(u_y' y_t + u_\delta' \delta_t \right) \right] = \frac{\mathbb{E}_{t-1}\left[\exp\left(\left(u_y + \Lambda_y \right)' y_t + \left(u_\delta + \tilde{\Lambda}_\delta \right)' \delta_t \right) \right]}{\mathbb{E}_{t-1}\left[\exp\left(\Lambda_y' y_t + \tilde{\Lambda}_\delta' \delta_t \right) \right]}$$

Using law of iterated expectations:

$$\begin{aligned}
\mathbb{E}_{t-1}^{\mathbb{Q}} \left[\exp\left(u_y' y_t + u_\delta' \delta_t \right) \right] &= \frac{\mathbb{E}_{t-1}\left[\exp\left(\left(u_y + \Lambda_y + \beta_\lambda \left[\frac{\mathsf{c}_\delta \odot \left(u_\delta + \tilde{\Lambda}_\delta \right)}{\mathbf{1} - \mathsf{c}_\delta \odot \left(u_\delta + \tilde{\Lambda}_\delta \right)} \right] \right)' y_t \right) \right]}{\mathbb{E}_{t-1}\left[\exp\left(\left(\Lambda_y + \beta_\lambda \left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta} \right] \right)' y_t \right) \right]} \\
&= \frac{\exp\left\{ \left(\frac{\mathsf{c}_y \odot \tilde{u}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y} \right)' \Phi_y y_{t-1} - \nu' \log\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y \right) \right\}}{\exp\left\{ \left(\frac{\mathsf{c}_y \odot \tilde{\Lambda}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y} \right)' \Phi_y y_{t-1} - \nu' \log\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y \right) \right\}} \\
&= \exp\left\{ \left(\frac{\mathsf{c}_y \odot \tilde{u}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y} - \frac{\mathsf{c}_y \odot \tilde{\Lambda}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y} \right)' \Phi_y y_{t-1} - \nu' \log\left(\frac{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y} \right) \right\}
\end{aligned}$$

where

$$\left\{ \begin{array}{rcl} \tilde{u}_y &=& u_y + \Lambda_y + \beta_\lambda \left[\frac{\mathsf{c}_\delta \odot \left(u_\delta + \tilde{\Lambda}_\delta \right)}{\mathbf{1} - \mathsf{c}_\delta \odot \left(u_\delta + \tilde{\Lambda}_\delta \right)} \right] \\ \tilde{\Lambda}_y &=& \Lambda_y + \beta_\lambda \left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta} \right] \end{array} \right.$$

Setting $u_\delta = \mathbf{0}$, we obtain:

$$\begin{aligned}
\frac{\mathsf{c}_y \odot \tilde{u}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y} - \frac{\mathsf{c}_y \odot \tilde{\Lambda}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y} &= \mathsf{c}_y \odot \frac{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot \tilde{u}_y - \tilde{\Lambda}_y \odot (\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y)}{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot (\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y)} \\
&= \mathsf{c}_y \odot \frac{\tilde{u}_y - \tilde{\Lambda}_y}{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot (\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y)} \\
&= \frac{\mathsf{c}_y \odot u_y}{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot (\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y)}
\end{aligned}$$

Second, looking at the term in the log:

$$\begin{aligned}
\frac{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y} &= \frac{\mathbf{1} - \mathsf{c}_y \odot \left[u_y + \Lambda_y + \operatorname{diag}\left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}\right] \beta_\lambda\right]}{\mathbf{1} - \mathsf{c}_y \odot \left[\Lambda_y + \operatorname{diag}\left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}\right] \beta_\lambda\right]} \\
&= \mathbf{1} - \frac{\mathsf{c}_y \odot u_y}{\mathbf{1} - \mathsf{c}_y \odot \left[\Lambda_y + \operatorname{diag}\left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}\right] \beta_\lambda\right]}
\end{aligned}$$

We set $\mathsf{c}_y^{\mathbb{Q}} = \frac{\mathsf{c}_y}{\mathbf{1} - \mathsf{c}_y \odot \left[\Lambda_y + \operatorname{diag}\left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}\right] \beta_\lambda\right]} = \frac{\mathsf{c}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y}$. We obtain:

$$\begin{aligned}
\mathbb{E}_{t-1}^{\mathbb{Q}}\left[\exp\left(u_y' y_t\right)\right] &= \exp\Bigg\{\left(\frac{\mathsf{c}_y \odot u_y}{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot (\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y)}\right)' \Phi_y y_{t-1} \\
&\quad - \nu' \log\left(\mathbf{1} - \frac{\mathsf{c}_y \odot u_y}{\mathbf{1} - \mathsf{c}_y \odot \left[\Lambda_y + \operatorname{diag}\left[\frac{\mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}{\mathbf{1} - \mathsf{c}_\delta \odot \tilde{\Lambda}_\delta}\right] \beta_\lambda\right]}\right)\Bigg\} \\
&= \exp\left\{\left(\frac{\mathsf{c}_y^{\mathbb{Q}} \odot u_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y}\right)' \Phi_y y_{t-1} - \nu' \log\left(\mathbf{1} - \mathsf{c}_y^{\mathbb{Q}} \odot u_y\right)\right\}
\end{aligned}$$

We then have:

$$\begin{aligned}
\frac{\mathsf{c}_y^{\mathbb{Q}} \odot u_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{u}_y} &= \mathsf{c}_y^{\mathbb{Q}} \odot \frac{u_y}{\mathbf{1} - \mathsf{c}_y \odot u_y - \mathsf{c}_y \odot \tilde{\Lambda}_y} \\
&= \mathsf{c}_y^{\mathbb{Q}} \odot \frac{u_y}{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot \left(\mathbf{1} - \frac{u_y \odot \mathsf{c}_y}{\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y}\right)} \\
&= \mathsf{c}_y^{\mathbb{Q}} \odot \frac{u_y}{\left(\mathbf{1} - \mathsf{c}_y \odot \tilde{\Lambda}_y\right) \odot \left(\mathbf{1} - u_y \odot \mathsf{c}_y^{\mathbb{Q}}\right)} \\
&= \frac{\mathsf{c}_y^{\mathbb{Q}} \odot u_y}{\mathbf{1} - u_y \odot \mathsf{c}_y^{\mathbb{Q}}} \odot \frac{\mathsf{c}_y^{\mathbb{Q}}}{\mathsf{c}_y}
\end{aligned}$$

So in the end we have:

$$\begin{aligned}
\mathbb{E}_{t-1}^{\mathbb{Q}}\left[\exp\left(u_y' y_t\right)\right] &= \exp\left\{\left(\frac{\mathsf{c}_y^{\mathbb{Q}} \odot u_y}{\mathbf{1} - u_y \odot \mathsf{c}_y^{\mathbb{Q}}} \odot \frac{\mathsf{c}_y^{\mathbb{Q}}}{\mathsf{c}_y}\right)' \Phi_y y_{t-1} - \nu \odot \log\left(\mathbf{1} - \mathsf{c}_y^{\mathbb{Q}} \odot u_y\right)\right\} \\
&= \exp\left\{\left(\frac{\mathsf{c}_y^{\mathbb{Q}} \odot u_y}{\mathbf{1} - u_y \odot \mathsf{c}_y^{\mathbb{Q}}}\right)' \operatorname{diag}\left(\frac{\mathsf{c}_y^{\mathbb{Q}}}{\mathsf{c}_y}\right) \Phi_y y_{t-1} - \nu \odot \log\left(\mathbf{1} - \mathsf{c}_y^{\mathbb{Q}} \odot u_y\right)\right\}.
\end{aligned}$$

and the result is obtained.

Since the classes of distributions are the same under the risk-neutral measure, w_t is an affine process under the risk-neutral measure and its conditional moment generating function is given by:

$$\begin{aligned}
\varphi_{w_t}^{\mathbb{Q}}(u) &:= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(u' w_{t+1}\right)\right] \\
&= \exp\left\{u_x'\left(\mu^{\mathbb{Q}} + \Phi_x^{\mathbb{Q}} x_t\right) + \frac{1}{2} u_x' \Sigma u_x + \left(\frac{\operatorname{diag}(\mathsf{c}_y^{\mathbb{Q}}) \widetilde{u}_y^{\mathbb{Q}}}{\mathbf{1} - \operatorname{diag}(\mathsf{c}_y^{\mathbb{Q}}) \widetilde{u}_y^{\mathbb{Q}}}\right)' \Phi_y^{\mathbb{Q}} y_t - \nu' \log\left[\mathbf{1} - \operatorname{diag}(\mathsf{c}_y^{\mathbb{Q}}) \widetilde{u}_y^{\mathbb{Q}}\right]\right\},
\end{aligned}$$

where

$$\widetilde{u}_y^{\mathbb{Q}} = \widetilde{u}_y = \left(\beta_\lambda^{\mathbb{Q}} \frac{\operatorname{diag}(\mathsf{c}_\delta^{\mathbb{Q}}) u_\delta}{\mathbf{1} - \operatorname{diag}(\mathsf{c}_\delta^{\mathbb{Q}}) u_\delta} + u_y\right).$$

Building on the property of affine processes, we have that the multi-horizon moment generating function of w_t is also an exponential-affine function of w_t under the risk-neutral measure. Let us introduce the following notation:

$$\varphi_{w_t}^{\mathbb{Q}}(u) = \exp\left(A^{\mathbb{Q}}(u) + B^{\mathbb{Q}}(u)' w_t\right).$$

We have that:

$$\varphi_{w_t}^{\mathbb{Q}}\left(u_1,\ldots,u_n\right) \quad := \quad \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\sum_{i=j}^{n} u_j'\, w_t\right)\right] = \exp\left[\mathcal{A}_n^{\mathbb{Q}}\left(u_1,\ldots,u_n\right) + \mathcal{B}_n^{\mathbb{Q}}\left(u_1,\ldots,u_n\right)'\, w_t\right],$$

where $\mathcal{A}_0^{\mathbb{Q}}\left(u_1,\ldots,u_n\right) = \mathbf{0}$ and $\mathcal{B}_n^{\mathbb{Q}}\left(u_1,\ldots,u_n\right) = \mathbf{0}$ and the loadings are defined through the following recursions:

$$\begin{aligned}
\mathcal{A}_n^{\mathbb{Q}}\left(u_1,\ldots,u_n\right) &= A^{\mathbb{Q}}\left(u_1 + \mathcal{B}_{n-1}^{\mathbb{Q}}\left(u_2,\ldots,u_n\right)\right) + \mathcal{A}_{n-1}^{\mathbb{Q}}\left(u_2,\ldots,u_n\right) \\
\mathcal{B}_n^{\mathbb{Q}}\left(u_1,\ldots,u_n\right) &= B^{\mathbb{Q}}\left(u_1 + \mathcal{B}_{n-1}^{\mathbb{Q}}\left(u_2,\ldots,u_n\right)\right).
\end{aligned}$$

This multi-horizon moment generating function will be calculated for all $n-1$ first arguments are equal, i.e. $u_1 = u_2 = \ldots = u_{n-1} = u$ and $u_n = v$. Thus, our notation $\mathcal{A}_n^{\mathbb{Q}}(u,v)$ and $\mathcal{B}_n^{\mathbb{Q}}(u,v)$ can be obtained through the above recursions by calculating $\mathcal{A}_n^{\mathbb{Q}}(u,\ldots,u,v)$ and $\mathcal{B}_n^{\mathbb{Q}}(u,\ldots,u,v)$.

H Pricing formulas for riskless nominal and real bonds

The price of riskless inflation-linked bonds and nominal bonds is respectively given by:

$$\begin{aligned}
D_t^{(n)^*} &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j}^*\right)\right] = e^{-n\kappa_0^{(r)}}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1} \kappa^{(r)'}\, w_{t+j}\right)\right] \\
D_t^{(n)} &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1}\left(r_{t+j}^* + \pi_{t+j+1}\right)\right)\right] \\
&= e^{-n\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}+\kappa^{(r)'}\, w_t\right)}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{n-1}\left(\kappa^{(r)}+\kappa^{(\pi)}\right)'\, w_{t+j} - \kappa^{(\pi)'}\, w_{t+n}\right)\right].
\end{aligned}$$

where $\kappa^{(r)} = \left(\kappa_x^{(r)'},\, \mathbf{0}_3',\, \mathbf{0}_2'\right)'$ and $\kappa^{(\pi)} = \left(\kappa_x^{(\pi)'},\, \kappa_y^{(\pi)'},\, 0,\, \kappa_\delta^{(\pi)},\, 0\right)'$. Thus, using our notation for the multi-horizon moment generating function of w_t under the risk-neutral measure, these expectations can be transformed as:

$$\begin{aligned}
D_t^{(n)^*} &= \exp\left\{-n\kappa_0^{(r)} + \mathcal{A}_n^{\mathbb{Q}}\left(-\kappa^{(r)}, \mathbf{0}\right) + \left[\mathcal{B}_n^{\mathbb{Q}}\left(-\kappa^{(r)}, \mathbf{0}\right) - \kappa^{(r)}\right]' w_t\right\} \\
D_t^{(n)} &= \exp\left\{-n\left(\kappa_0^{(r)} + \kappa_0^{(\pi)}\right) + \mathcal{A}_n^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)}, -\kappa^{(\pi)}\right) + \left[\mathcal{B}_n^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)}, -\kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\right\}.
\end{aligned}$$

I Pricing formulas for nominal treasuries and TIPS

Let us first focus on nominal bonds. We rewrite Equation (79) for convenience:

$$\begin{aligned}
B_t^{(n)} &= \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)} - \sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*} + \pi_{t+j+1}\right)\right) \times \left(\mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)} = 0\right\} - \mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)} = 0\right\}\right)\right] \\
&+ \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1}\left(r_{t+j}^{(1)^*} + \pi_{t+j+1}\right)\right)\mathbb{1}\left\{\sum_{j=0}^{n}\delta_{t+j}^{(c)} = 0\right\}\right].
\end{aligned}$$

We define $\delta_t^{(c)} = \mathbf{e}_c' w_t$. Focusing on the first indicator term in the above equation, we can write:

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)} - \sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*} + \pi_{t+j+1}\right)\right) \times \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)} = 0\right\}\right] \\
&= e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)-\kappa^{(r)\prime}w_t}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{i-1}\left(\kappa^{(r)} + \kappa^{(\pi)}\right)' w_{t+j} - \left(\mathbf{e}_c + \kappa^{(\pi)}\right)' w_{t+i}\right) \times \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)} = 0\right\}\right].
\end{aligned}$$

Using the lemma 3.1 of Monfort et al. (2020), we have:

$$\begin{aligned}
&e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)-\kappa^{(r)\prime}w_t}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{i-1}\left(\kappa^{(r)} + \kappa^{(\pi)}\right)' w_{t+j} - \left(\mathbf{e}_c + \kappa^{(\pi)}\right)' w_{t+i}\right) \times \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)} = 0\right\}\right] \\
&= \lim_{\mathsf{u}\to+\infty} e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)-\kappa^{(r)\prime}w_t}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{i-1}\left[\left(\kappa^{(r)} + \kappa^{(\pi)}\right)' w_{t+j} + \mathsf{u}\delta_{t+j}^{(c)}\right] - \left(\mathbf{e}_c + \kappa^{(\pi)}\right)' w_{t+i}\right)\right] \\
&= \lim_{\mathsf{u}\to+\infty} e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)-\kappa^{(r)\prime}w_t}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{i-1}\left(\kappa^{(r)} + \kappa^{(\pi)} + \mathsf{u}\mathbf{e}_c\right)' w_{t+j} - \left(\mathbf{e}_c + \kappa^{(\pi)}\right)' w_{t+i}\right)\right] \\
&= \lim_{\mathsf{u}\to+\infty} \exp\Big\{-i\left(\kappa_0^{(r)} + \kappa_0^{(\pi)}\right) + \mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c,\ -\mathbf{e}_c - \kappa^{(\pi)}\right) \\
&\qquad + \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c,\ -\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\Big\}
\end{aligned}$$

Applying the same logic to the remaining terms, and assuming default has not occurred at date t, we obtain the result of Equation (81):

$$
\begin{aligned}
B_t^{(n)} = & \lim_{\mathsf{u}\to+\infty} e^{-\kappa^{(r)\prime} w_t} \sum_{i=1}^{n} e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)} \times \Bigg[\\
& \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\, -\mathbf{e}_c-\kappa^{(\pi)}\right)+\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\, -\mathbf{e}_c-\kappa^{(\pi)}\right)' w_t\right\} \\
- & \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\, -\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)+\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\, -\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)' w_t\right\}\Bigg] \\
& + \exp\Big\{-n\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)+\mathcal{A}_n^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\, -\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right) \\
& \qquad + \mathcal{B}_n^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\, -\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)' w_t\Big\}.
\end{aligned}
$$

Let us now turn to TIPS valuation. Again, for convenience, we rewrite the general pricing formula given by Equation (78) below:

$$
\begin{aligned}
B_t^{(n)*} = & \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)}-\sum_{j=0}^{i-1}\left[r_{t+j}^{(1)*}+(1-\rho^*)\pi_{t+j+1}\right]\right)\cdot \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}\mathbb{1}\left\{\delta_{t+i}^{(c)}>0\right\}\right] \\
& + \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(\ell)}-\sum_{j=0}^{i-1}r_{t+j}^{(1)*}\right)\cdot \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}\mathbb{1}\left\{\delta_{t+i}^{(c)}=0\cap\delta_{t+i}^{(\ell)}>0\right\}\right] \\
& + \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1}r_{t+j}^{(1)*}\right)\mathbb{1}\left\{\sum_{j=0}^{n}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}\right] \\
= & \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}}\Bigg[\exp\left(-\delta_{t+i}^{(c)}-\sum_{j=0}^{i-1}\left[r_{t+j}^{(1)*}+(1-\rho^*)\pi_{t+j+1}\right]\right)\times \\
& \qquad \left(\mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}-\mathbb{1}\left\{\sum_{j=0}^{i-1}\left(\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}\right)+\delta_{t+i}^{(c)}=0\right\}\right)\Bigg] \\
& + \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(\ell)}-\sum_{j=0}^{i-1}r_{t+j}^{(1)*}\right)\cdot\left(\mathbb{1}\left\{\sum_{j=0}^{i-1}\left(\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}\right)+\delta_{t+i}^{(c)}=0\right\}-\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}\right)\right] \\
& + \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1}r_{t+j}^{(1)*}\right)\mathbb{1}\left\{\sum_{j=0}^{n}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}\right]
\end{aligned}
$$

We define $\delta_t^{(\ell)} = \mathbf{e}_\ell' w_t$. Assuming no default or liquidity event at date t, the first term of this equation can be detailed as:

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)} - \sum_{j=0}^{i-1}\left[r_{t+j}^{(1)^*} + (1-\rho^*)\pi_{t+j+1}\right]\right) \times \mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)} + \delta_{t+j}^{(\ell)} = 0\right\}\right] \\
= & \lim_{\mathsf{u}\to+\infty} e^{-i\left[\kappa_0^{(r)}+(1-\rho^*)\kappa_0^{(\pi)}\right]-\kappa^{(r)\prime}w_t} \\
& \times \quad \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{i-1}\left(\kappa^{(r)} + (1-\rho^*)\kappa^{(\pi)} + \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right)\right)' w_{t+j} - \left(\mathbf{e}_c + (1-\rho^*)\kappa^{(\pi)}\right)' w_{t+i}\right)\right] \\
= & \lim_{\mathsf{u}\to+\infty} \exp\left\{-i\left[\kappa_0^{(r)} + (1-\rho^*)\kappa_0^{(\pi)}\right] + \mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right), -\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\right)\right. \\
& + \quad \left.\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right), -\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\right\}
\end{aligned}
$$

Using the same properties, we obtain:

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)} - \sum_{j=0}^{i-1}\left[r_{t+j}^{(1)^*} + (1-\rho^*)\pi_{t+j+1}\right]\right) \times \mathbb{1}\left\{\sum_{j=0}^{i-1}\left(\delta_{t+j}^{(c)} + \delta_{t+j}^{(\ell)}\right) + \delta_{t+i}^{(c)} = 0\right\}\right] \\
= & \lim_{\mathsf{u}\to+\infty} \exp\left\{-i\left[\kappa_0^{(r)} + (1-\rho^*)\kappa_0^{(\pi)}\right] + \mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right), -\mathsf{u}\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\right)\right. \\
& + \quad \left.\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right), -\mathsf{u}\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\right\},
\end{aligned}
$$

for the liquidity event:

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(\ell)} - \sum_{j=0}^{i-1} r_{t+j}^{(1)^*}\right) \cdot \mathbb{1}\left\{\sum_{j=0}^{i-1}\left(\delta_{t+j}^{(c)} + \delta_{t+j}^{(\ell)}\right) + \delta_{t+i}^{(c)} = 0\right\}\right] \\
= & \lim_{\mathsf{u}\to+\infty} \exp\left\{-i\kappa_0^{(r)} + \mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right), -\mathsf{u}\mathbf{e}_c - \mathbf{e}_\ell\right)\right. \\
& + \quad \left.\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \mathsf{u}\left(\mathbf{e}_c + \mathbf{e}_\ell\right), -\mathsf{u}\mathbf{e}_c - \mathbf{e}_\ell\right) - \kappa^{(r)}\right]' w_t\right\},
\end{aligned}
$$

and

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(\ell)}-\sum_{j=0}^{i-1}r_{t+j}^{(1)^*}\right)\cdot\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}+\delta_{t+j}^{(\ell)}=0\right\}\right]\\
=&\lim_{\mathsf{u}\to+\infty}\exp\left\{-i\kappa_0^{(r)}+\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\mathsf{u}\left(\mathbf{e}_c+\mathbf{e}_\ell\right),-\mathsf{u}\left(\mathbf{e}_c+\mathbf{e}_\ell\right)\right)\right.\\
&\left.+\quad\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\mathsf{u}\left(\mathbf{e}_c+\mathbf{e}_\ell\right),-\mathsf{u}\left(\mathbf{e}_c+\mathbf{e}_\ell\right)\right)-\kappa^{(r)}\right]'w_t\right\},
\end{aligned}$$

which is also the last term when $i=n$. Putting all these terms together, we obtain the result of Equation (80).

J Pricing formulas for CDS spreads

The protection buyer value is given by:

$$\mathrm{PB}_t^{(n)}=\mathcal{S}_t^{(n)}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*}+\pi_{t+j+1}\right)\right)\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}=0\right\}\right]$$

Applying the same pricing principle as in Appendix I, we can easily express the protection buyer value as:

$$\begin{aligned}
&\mathcal{S}_t^{(n)}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*}+\pi_{t+j+1}\right)\right)\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}=0\right\}\right]\\
=&\lim_{\mathsf{u}\to+\infty}\mathcal{S}_t^{(n)}\sum_{i=1}^{n}e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)-\kappa^{(r)'}w_t}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{i-1}\left(\kappa^{(r)}+\kappa^{(\pi)}+\mathsf{u}\mathbf{e}_c\right)'w_{t+j}-\left(\mathsf{u}\mathbf{e}_c+\kappa^{(\pi)}\right)'w_{t+i}\right)\right]\\
=&\lim_{\mathsf{u}\to+\infty}\mathcal{S}_t^{(n)}e^{-\kappa^{(r)'}w_t}\sum_{i=1}^{n}\exp\left\{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)+\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c\right)\right.\\
&\left.+\quad\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c\right)'w_t\right\}.
\end{aligned}$$

The protection seller value is given by:

$$\mathrm{PS}_t(n)=\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{i-1}\left(r_{t+j}^{(1)^*}+\pi_{t+j+1}\right)\right)\left(1-e^{-\delta_{t+i}^{(c)}}\right)\left(\mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}=0\right\}-\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}=0\right\}\right)\right]$$

We can separate the term in $\left(1-e^{-\delta_{t+i}^{(c)}}\right)$ in two and treat these two terms. The first term,

$$\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{i-1}\left(r_{t+j}^*+\pi_{t+j+1}\right)\right)\left(\mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}=0\right\}-\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}=0\right\}\right)\right],$$

would be the price of a nominal treasury with recovery payment of the full face value, forgetting the principal repayment at maturity provided no default has happened (the last term of Equation (79) is missing). The second term,

$$\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\delta_{t+i}^{(c)}-\sum_{j=0}^{i-1}\left(r_{t+j}^*+\pi_{t+j+1}\right)\right)\left(\mathbb{1}\left\{\sum_{j=0}^{i-1}\delta_{t+j}^{(c)}=0\right\}-\mathbb{1}\left\{\sum_{j=0}^{i}\delta_{t+j}^{(c)}=0\right\}\right)\right],$$

is exactly the first row of Equation (79), so it is the price of a nominal treasury, forgetting the principal repayment at maturity provided no default has happened. In the end, taking the difference between these two terms, it is innocuous to add the discounted value of the last payment in both terms since they are canceling out in the difference. We hence obtain that the protection seller value is the difference between the price of a nominal treasury with recovery payment of \$1 and the price of the standard nominal treasury. The result of Equation (83) is obtained by equation the protection buyer and seller values.

K Gradient computation for measurement equations

We use the extended Kalman filter for estimation, which requires the computation of the gradient of the pricing equations in the factors. Since our pricing equations are closed-form, we have closed-form gradients as well. Since these computations are the result of tedious algebra, we only present the results without justification.

Let us start with riskless yields. Given the formulation of the ILS and nominal riskless yields (see Equations 76-77), we trivially have:

$$\begin{aligned}\frac{\partial\,\mathrm{ILS}_t^{(n)}}{\partial\,w_t} &= \left(\mathsf{b}_{ils,n}^{(x)\prime},\,\mathsf{b}_{ils,n}^{(y)\prime},\,\mathbf{0}_2'\right)'\\ \frac{\partial\,r_t^{(n)}}{\partial\,w_t} &= \left(\mathsf{b}_{rf,n}^{(x)\prime},\,\mathsf{b}_{rf,n}^{(y)\prime},\,\mathbf{0}_2'\right)'.\end{aligned}$$

Let us turn now to nominal treasuries and TIPS. Continuously compounded yields of these bonds are respectively denoted by $R_t^{(n)} = -n^{-1} \log B_t^{(n)}$ and $R_t^{(n)^*} = -n^{-1} \log B_t^{(n)^*}$. It is useful to define the differentials with respect to the price instead of the yield directly. Using the chain rule, we have:

$$\begin{aligned}
\frac{\partial R_t^{(n)}}{\partial w_t} &= -\frac{1}{n} \times \frac{\partial B_t^{(n)}}{\partial w_t} \times \frac{1}{B_t^{(n)}} \\
\frac{\partial R_t^{(n)^*}}{\partial w_t} &= -\frac{1}{n} \times \frac{\partial B_t^{(n)^*}}{\partial w_t} \times \frac{1}{B_t^{(n)^*}} .
\end{aligned}$$

Let us focus on the differential of the nominal bond price first.

$$\begin{aligned}
\frac{\partial B_t^{(n)}}{\partial w_t} = \lim_{\mathsf{u}\to+\infty} \sum_{i=1}^{n} e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)} \Bigg[& \\
\exp\Big\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathbf{e}_c - \kappa^{(\pi)}\right) &+ \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\Big\} \\
&\times \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right] \\
- \exp\Big\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathsf{u}\mathbf{e}_c - \kappa^{(\pi)}\right) &+ \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathsf{u}\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\Big\} \\
&\times \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathsf{u}\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right]\Bigg] \\
+ \exp\Big\{-n\left(\kappa_0^{(r)} + \kappa_0^{(\pi)}\right) &+ \mathcal{A}_n^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathsf{u}\mathbf{e}_c - \kappa^{(\pi)}\right) \\
&+ \left[\mathcal{B}_n^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathsf{u}\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right]' w_t\Big\} \\
&\times \left[\mathcal{B}_n^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c, -\mathsf{u}\mathbf{e}_c - \kappa^{(\pi)}\right) - \kappa^{(r)}\right] .
\end{aligned}$$

For TIPS, applying a similar principle:

$$
\begin{aligned}
\frac{\partial B_t^{(n)^*}}{\partial w_t} = \lim_{\mathsf{u}\to+\infty} \quad & \sum_{i=1}^{n} e^{-i\left[\kappa_0^{(r)}\right]} \Bigg[\\
& \exp\Big\{-i\,(1-\rho^*)\,\kappa_0^{(\pi)} + \mathcal{A}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\Big) \\
& \qquad + \quad \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\Big) - \kappa^{(r)}\Big]' \, w_t\Big\} \\
& \qquad\qquad \times \quad \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\Big) - \kappa^{(r)}\Big] \\
- \quad & \exp\Big\{-i\,(1-\rho^*)\,\kappa_0^{(\pi)} + \mathcal{A}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\Big) \\
& \qquad + \quad \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\Big) - \kappa^{(r)}\Big]' \, w_t\Big\} \\
& \qquad\qquad \times \quad \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - (1-\rho^*)\kappa^{(\pi)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}\mathbf{e}_c - (1-\rho^*)\kappa^{(\pi)}\Big) - \kappa^{(r)}\Big] \\
+ \quad & \exp\Big\{\mathcal{A}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}\mathbf{e}_c - \mathbf{e}_\ell\Big) + \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}\mathbf{e}_c - \mathbf{e}_\ell\Big) - \kappa^{(r)}\Big]' \, w_t\Big\} \\
& \qquad\qquad \times \quad \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}\mathbf{e}_c - \mathbf{e}_\ell\Big) - \kappa^{(r)}\Big] \\
- \quad & \exp\Big\{\mathcal{A}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell)\Big) + \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell)\Big) - \kappa^{(r)}\Big]' \, w_t\Big\} \\
& \qquad\qquad \times \quad \Big[\mathcal{B}_i^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell)\Big) - \kappa^{(r)}\Big]\Bigg] \\
+ \quad & \exp\Big\{-n\kappa_0^{(r)} + \mathcal{A}_n^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell)\Big) + \Big[\mathcal{B}_n^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell)\Big) - \kappa^{(r)}\Big]' \, w_t\Big\} \\
& \qquad\qquad \times \quad \Big[\mathcal{B}_n^{\mathbb{Q}}\Big(-\kappa^{(r)} - \mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell),\, -\mathsf{u}(\mathbf{e}_c+\mathbf{e}_\ell)\Big) - \kappa^{(r)}\Big]\,.
\end{aligned}
$$

Last, for the CDS, let us denote by:

$$\mathcal{S}_t^{(n)} = \frac{f_n(w_t)}{g_n(w_t)}\,,$$

where $f_n(\bullet)$ and $g_n(\bullet)$ are explicit functions given by Equation (83). Using differentiation rules:

$$\frac{\partial\,\mathcal{S}_t^{(n)}}{\partial\,w_t} = \frac{1}{g_n(w_t)} \times \frac{\partial\,f_n(w_t)}{\partial\,w_t} - \frac{f_n(w_t)}{g_n(w_t)^2} \times \frac{\partial\,g_n(w_t)}{\partial\,w_t}\,.$$

The differential of $f_n(w_t)$ is easily obtained as a function of the differential of nominal treasuries:

$$
\begin{aligned}
\frac{\partial f_n(w_t)}{\partial w_t} = & \lim_{\mathsf{u}\to+\infty}\sum_{i=1}^{n} e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)}\Bigg[\\
& \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\kappa^{(\pi)}\right)+\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]' w_t\right\} \\
& \qquad \times \quad \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\kappa^{(\pi)}\right)-\kappa^{(r)}\right] \\
- & \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)+\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]' w_t\right\} \\
& \qquad \times \quad \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right] \\
- & \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathbf{e}_c-\kappa^{(\pi)}\right)+\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]' w_t\right\} \\
& \qquad \times \quad \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right] \\
+ & \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)+\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]' w_t\right\} \\
& \qquad \times \quad \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathsf{u}\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]\Bigg] \\
= & \lim_{\mathsf{u}\to+\infty}\sum_{i=1}^{n} e^{-i\left(\kappa_0^{(r)}+\kappa_0^{(\pi)}\right)}\Bigg[\\
& \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\kappa^{(\pi)}\right)+\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]' w_t\right\} \\
& \qquad \times \quad \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\kappa^{(\pi)}\right)-\kappa^{(r)}\right] \\
- & \exp\left\{\mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathbf{e}_c-\kappa^{(\pi)}\right)+\left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]' w_t\right\} \\
& \qquad \times \quad \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)}-\kappa^{(\pi)}-\mathsf{u}\mathbf{e}_c,\,-\mathbf{e}_c-\kappa^{(\pi)}\right)-\kappa^{(r)}\right]\Bigg].
\end{aligned}
$$

and, for the function $g_n(w_t)$:

$$\frac{\partial\, g_n(w_t)}{\partial\, w_t} = \lim_{\mathsf{u}\to+\infty} \sum_{i=1}^{n} \Bigg(\exp\Big\{ -i\left(\kappa_0^{(r)} + \kappa_0^{(\pi)}\right) + \mathcal{A}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c,\ -\kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c\right) + \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c,\ -\kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c\right) - \kappa^{(r)}\right]' w_t \Big\} \times \left[\mathcal{B}_i^{\mathbb{Q}}\left(-\kappa^{(r)} - \kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c,\ -\kappa^{(\pi)} - \mathsf{u}\mathbf{e}_c\right) - \kappa^{(r)}\right] \Bigg).$$

L Identification constraints

For econometric identification, $\mathsf{c}_y^{\mathbb{Q}} = \mathbf{1}$ imposed. For parsimony, the covariance matrix of measurement errors is assumed to be diagonal, and each block of observables has a different standard deviation parameter. The standard deviation of the liquidity proxy measurement errors is set to a fifth of its in-sample standard deviation, as it provides a reasonable fit of the proxy. The standard deviation of ILSBEI measurement errors lies below 6bps. This constraint is not binding at the optimum. Last, the measurement errors on the CDS term structure depend on a CDS illiquidity index. The CDS illiquidity is measured as the inverse of the aggregated monthly depth on the 5y U.S. sovereign CDS, computed as the number of contributors whose contributions were included in the final composite value (Markit data, see Figure A6, bottom panel). This measure is inspired by Qiu and Yu (2012).[11] The final standard deviations of CDS measurement errors are obtained by scaling the series by an estimated positive parameter.

[11] Alternative liquidity measures exist in the literature but do not cover the sample that we consider in the paper (see e.g. Wang et al. (2021) or the references in Augustin et al. (2014)).

M Supplementary Tables and Figures

Table A1: First Difference Regressions: Post-Crisis

Table A1 presents the results of regressions of the first difference in the five-year ILSBEI spread on the growth in Treasury debt held by the public (G) and the first difference in of the Euro-denominated five-year U.S. Treasury CDS spread (CDS) with controls for liquidity and slow moving capital. Liquidity and slow-moving capital variables are the VIX index (VIX), the spread between LIBOR and OIS ($L-OIS$), the difference between an off-the-run and on-the-run 10-year nominal Treasury security (OTR), and the noise measure of Hu, Pan and Wang (2013) (HPW). Standard errors are presented in parentheses and corrected for autocorrelation and heteroskedasticity via Newey-West with three lags. Data are sampled at the monthly frequency and cover the period January, 2010 through December, 2020.

Specification	(1)	(2)	(3)	(4)	(5)
G	−0.004		−0.004		−0.000
SE	(0.004)		(0.004)		(0.005)
CDS		0.067	0.061		0.041
SE		(0.106)	(0.106)		(0.110)
VIX				0.001	0.001
SE				(0.001)	(0.001)
$L-OIS$				0.058	0.055
SE				(0.037)	(0.038)
OTR				−0.011	−0.002
SE				(0.174)	(0.181)
HPW				0.012	0.013
SE				(0.010)	(0.010)
R^2	0.007	0.003	0.009	0.066	0.067

Table A2: First Stage Instrumental Variables Regressions: First Difference

Table A2 presents the results of regressions of debt growth, the first difference in CDS spreads, and the first difference in liquidity controls. Data are sampled at the monthly frequency over June, 2005 through December, 2020.

	VIX	$L-OIS$	OTR	HPW	CDS
G	0.314	0.018*	0.009***	0.183***	0.002
SE	(0.283)	(0.010)	(0.003)	(0.049)	(0.003)
VIX					0.001
SE					(0.001)
$L-OIS$					0.007
SE					(0.024)
OTR					−0.071
SE					(0.115)
HPW					0.007
SE					(0.006)
R^2	0.007	0.016	0.055	0.069	0.032

Table A3: Decomposition of ILSBEI Regressions

Table A3 presents decomposed versions of the results of regressions of the five-year ILSBEI spread on the Euro-denominated five-year U.S. Treasury CDS spread (CDS), liquidity and slow-moving capital controls, and growth in aggregate Treasury debt held by the public (G) presented in Table 2. Liquidity and slow-moving capital controls are the level of the VIX index (VIX), the spread between LIBOR and OIS ($L{-}OIS$), the spread between an off-the-run and on-the-run 10-year Treasury (OTR), and the noise measure of Hu, Pan, and Wang (HPW). Column (1) presents results for the ILS, column (2) for the ILS minus nominal Treasury (TSY), column (3) for the breakeven inflation (BEI), column (4) for the nominal Treasury, and column (5) for the inflation-protected Treasury (TIPS). Standard errors in parentheses are corrected for heteroskedasticity via Newey-West with three lags. Data are sampled at the monthly frequency and cover the period July, 2005 through December, 2020.

	ILS	ILS-TSY	BEI	TSY	TIPS
G	−0.011	0.003	−0.015*	−0.014	0.001
SE	(0.008)	(0.021)	(0.008)	(0.022)	(0.022)
CDS	−0.848**	4.390***	−0.573*	−5.238***	4.665***
SE	(0.383)	(0.758)	(0.337)	(0.915)	(0.822)
VIX	−0.026***	0.009	−0.026***	−0.035**	−0.009
SE	(0.009)	(0.010)	(0.008)	(0.015)	(0.011)
$L-OIS$	0.228	0.653**	0.088	−0.426	−0.514
SE	(0.205)	(0.324)	(0.191)	(0.436)	(0.034)
OTR 2.634***	−0.808	2.453***	3.441**	0.989	
SE	(0.531)	(1.676)	(0.563)	(1.664)	(1.762)
HPW	−0.138***	−0.244***	−0.193***	0.106	0.299***
SE	(0.043)	(0.064)	(0.042)	(0.069)	(0.065)
R^2	0.450	0.522	0.665	0.512	0.564

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Table A4: Decomposition of ILSBEI Regressions: Subsample

Table A4 presents decomposed versions of the results of regressions of the five-year ILSBEI spread on the Euro-denominated five-year U.S. Treasury CDS spread (CDS), liquidity and slow-moving capital controls, and growth in aggregate Treasury debt held by the public (G) presented in Table 2. Liquidity and slow-moving capital controls are the level of the VIX index (VIX), the spread between LIBOR and OIS ($L-OIS$), the spread between an off-the-run and on-the-run 10-year Treasury (OTR), and the noise measure of Hu, Pan, and Wang (HPW). Column (1) presents results for the ILS, column (2) for the ILS minus nominal Treasury (TSY), column (3) for the breakeven inflation (BEI), column (4) for the nominal Treasury, and column (5) for the inflation-protected Treasury (TIPS). Standard errors in parentheses are corrected for heteroskedasticity via Newey-West with three lags. Data are sampled at the monthly frequency and cover the period January, 2010 through December, 2020.

	ILS	ILS-TSY	BEI	TSY	TIPS
G	0.004	0.017	0.002	−0.013	−0.015
SE	(0.005)	(0.014)	(0.006)	(0.015)	(0.014)
CDS	1.337***	2.071***	1.046***	−0.735	−1.780**
SE	(0.341)	(0.761)	(0.353)	(0.770)	(0.791)
VIX	−0.027***	0.013	−0.027***	−0.039***	−0.012
SE	(0.005)	(0.009)	(0.005)	(0.011)	(0.009)
$L-OIS$	−0.071	−0.669	−0.083	0.598	0.681
SE	(0.314)	(0.440)	(0.328)	(0.617)	(0.434)
OTR 0.913	−0.005	1.042*	0.918	−0.124	
SE	(0.557)	(1.271)	(0.604)	(1.377)	(1.256)
HPW	−0.199***	−0.480***	−0.233***	0.281**	0.514***
SE	(0.066)	(0.111)	(0.067)	(0.111)	(0.115)
R^2	0.538	0.366	0.513	0.271	0.343

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Table A5: ILSBEI Regressions: Alternate Tenors

Table A6 presents the results of regressions of the ILSBEI spread on debt growth, CDS spreads and liquidity controls. Results for the two-year, three-year, seven-year, and ten-year tenors are presented in columns 1 - 4, respectively. Data are sampled at the monthly frequency over June, 2005 through December, 2020. First and second stage parameters are simultaneously estimated via single-stage GMM with Newey-West-corrected standard errors.

	2-Year	3-Year	7-Year	10-Year
$\widehat{CDS}$	1.097**	1.324***	0.868**	0.385
SE	(0.521)	(0.511)	(0.341)	(0.250)
VIX	−0.002	−0.002	0.003	0.001
SE	(0.003)	(0.002)	(0.002)	(0.001)
$L-OIS$	0.218*	0.138**	0.002	−0.097**
SE	(0.115)	(0.068)	(0.046)	(0.049)
OTR	−0.381	0.053	−0.193	−0.636***
SE	(0.298)	(0.182)	(0.124)	(0.129)
HPW	0.072***	0.070***	0.057***	0.064***
SE	(0.026)	(0.012)	(0.008)	(0.009)
R^2	0.608	0.811	0.698	0.572

Table A6: ILSBEI Regressions: Alternate Tenors, post-Crisis

Table A6 presents the results of regressions of the ILSBEI spread on debt growth, CDS spreads and liquidity controls. Results for the two-year, three-year, seven-year, and ten-year tenors are presented in columns 1 - 4, respectively. Data are sampled at the monthly frequency over January, 2010 through December, 2020. First and second stage parameters are simultaneously estimated via single-stage GMM with Newey-West-corrected standard errors.

	2-Year	3-Year	7-Year	10-Year
G	0.001	0.002	0.001	−0.001
SE	(0.003)	(0.002)	(0.001)	(0.001)
CDS	0.013	0.009	0.316***	0.068
SE	(0.184)	(0.121)	(0.067)	(0.065)
VIX	0.000	−0.000	0.001	0.001
SE	(0.003)	(0.002)	(0.001)	(0.001)
$L-OIS$	0.014	0.026	−0.020	0.009
SE	(0.141)	(0.329)	(0.044)	(0.053)
OTR	−0.368	0.050	−0.338**	−0.505***
SE	(0.354)	(0.235)	(0.144)	(0.111)
HPW	0.050	0.050***	0.030*	0.049***
SE	(0.031)	(0.019)	(0.016)	(0.011)
R^2	0.030	0.184	0.313	0.233

Table A7: Instrumental Variables Regressions: Post-Crisis

Table A7 presents the results of instrumental variables regressions of the ILSBEI spread on CDS spreads and liquidity controls. In the first stage, we regress the liquidity and slow-moving capital controls on the growth in Treasury debt and regress the CDS spread on orthogonalized liquidity and slow-moving capital controls (where orthogonalization is denoted by superscript $^{\perp}$) and growth in Treasury debt. Results are presented in Panel A. In the second stage, we regress the ILSBEI on the predicted CDS spread and the orthogonalized controls, and present results in column (1) of Panel B. We repeat the second stage regressions for the ILS in column (2), the nominal Treasury in column (3), and TIPS in column (4). Data are sampled at the monthly frequency over January, 2010 through December, 2020. First and second stage parameters are simultaneously estimated via single-stage GMM with Newey-West-corrected standard errors.

Panel A : First Stage Regressions

	VIX	$L-OIS$	OTR	HPW	CDS
G	0.540***	−0.002	0.006**	0.025	0.009**
SE	(0.094)	(0.002)	(0.003)	(0.024)	(0.004)
$VIX^{\perp}$					−0.002
SE					(0.003)
$L-OIS^{\perp}$					0.181
SE					(0.114)
$OTR^{\perp}$					1.095***
SE					(0.240)
$HPW^{\perp}$					−0.055***
SE					(0.021)
R^2	0.259	0.011	0.240	0.063	0.523

Panel B: Second Stage Regressions

	ILSBEI	ILS	ILS-TSY	BEI	TSY	TIPS
$\widehat{CDS}$	0.536***	0.260	3.567	−0.276	−3.306	−3.031
SE	(0.163)	(0.811)	(2.874)	(0.915)	(3.225)	(2.833)
$VIX^{\perp}$	0.000	−0.029***	0.009	−0.029***	−0.038***	−0.009
SE	(0.002)	(0.008)	(0.010)	(0.007)	(0.011)	(0.010)
$L-OIS^{\perp}$	0.065	0.171	−0.294	0.106	0.465	0.359
SE	(0.062)	(0.404)	(0.516)	(0.394)	(0.649)	(0.492)
$OTR^{\perp}$	0.189	2.376***	2.263*	2.187***	0.113	−2.074*
SE	(0.157)	(0.604)	(1.163)	(0.592)	(1.280)	(1.134)
$HPW^{\perp}$	0.018	−0.272***	−0.594***	−0.290***	0.322**	0.612***
SE	(0.017)	(0.079)	(0.147)	(0.088)	(0.132)	(0.149)
R^2	0.267	0.442	0.300	0.456	0.263	0.294

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

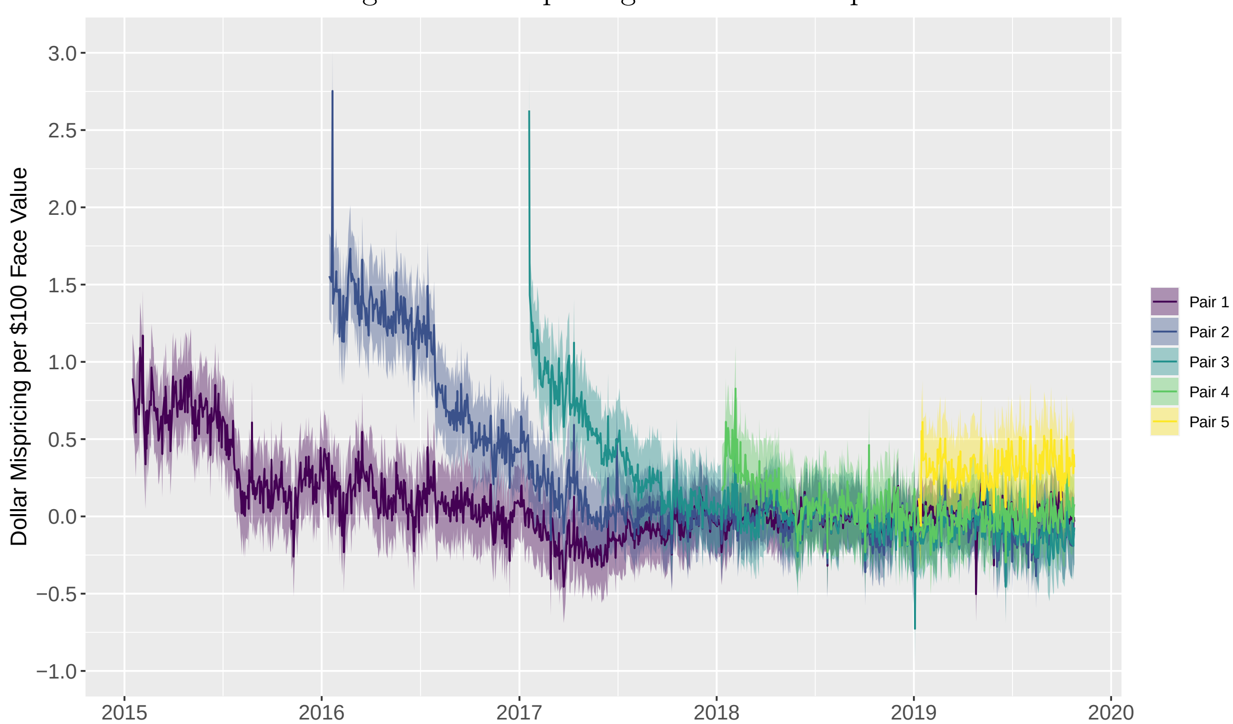
Table A8: TIPS pairs with the same maturity date

CUSIP	Issuance date	Maturity date	Coupon rate
912810FR	2004-07-15	2025-01-15	2.375%
912828H4	2015-01-15		0.25%
912810FS	2006-01-15	2026-01-15	2%
912828N7	2016-01-29		0.625%
912810PS	2007-01-15	2027-01-15	2.375%
912828V4	2017-01-15		0.375%
912810PV	2008-01-15	2028-01-15	1.75%
9128283R	2018-01-15		0.5%
912810PZ	2009-01-15	2029-01-15	2.5%
9128285W	2019-01-15		0.875%

Table A9: Real cashflows of treasuries in the term structure model

	Timeline	$t+\tau$	$t+n$
No default	Nominal	0	$e^{-\pi_{t+1}-\ldots-\pi_{t+n}}$
	TIPS	0	1
Default at $t+\tau$	Nominal	$e^{-\delta^{(c)}_{t+\tau}-\pi_{t+1}-\ldots-\pi_{t+\tau}}$	0
	TIPS	$e^{-\delta^{(c)}_{t+\tau}+(\rho^*-1)(\pi_{t+1}+\ldots+\pi_{t+\tau})}$	0
Liquidity at $t+\tau$	Nominal	0	$e^{-\pi_{t+1}-\ldots-\pi_{t+n}}$
	TIPS	$e^{-\delta^{(\ell)}_{t+\tau}}$	0

Figure A1: Mispricing between TIPS pairs



Note: These series present the mispricing between TIPS that have the same maturity date but different issuance dates. Each series is the price of a long-short portfolio of the two TIPS in the pair, and a short position in a zero-coupon TIPS of the same time to maturity. The weights are computed such that the aggregated portfolio yields zero cash flows if there is no default event. To compute the price of the zero-coupon TIPS, we use the parameters of a Nelson-Siegel-Svensson curve fitted on a daily basis on TIPS coupon bonds. To include errors resulting from smoothing the curve, we include confidence bands corresponding to plus or minus 3bps on the smoothed zero-coupon yield.

Figure A2: 6m OIS, inflation and liquidity proxy fitted values

The model is estimated with extended Kalman filter. Data range from November 2004 to December 2019. The black solid line presents the observation data used as input for estimation. The grey dashed line presents the fitted values produced through the filtered factors presented on Figure 5.

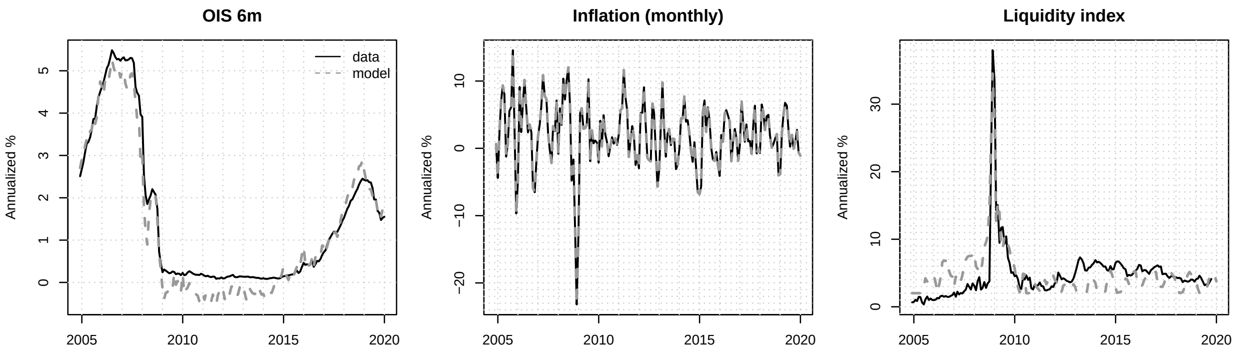


Figure A3: Inflation-linked swaps fitted values

The model is estimated with extended Kalman filter. Data range from November 2004 to December 2019. The black solid line presents the observation data used as input for estimation. The grey dashed line presents the fitted values produced through the filtered factors presented on Figure 5.

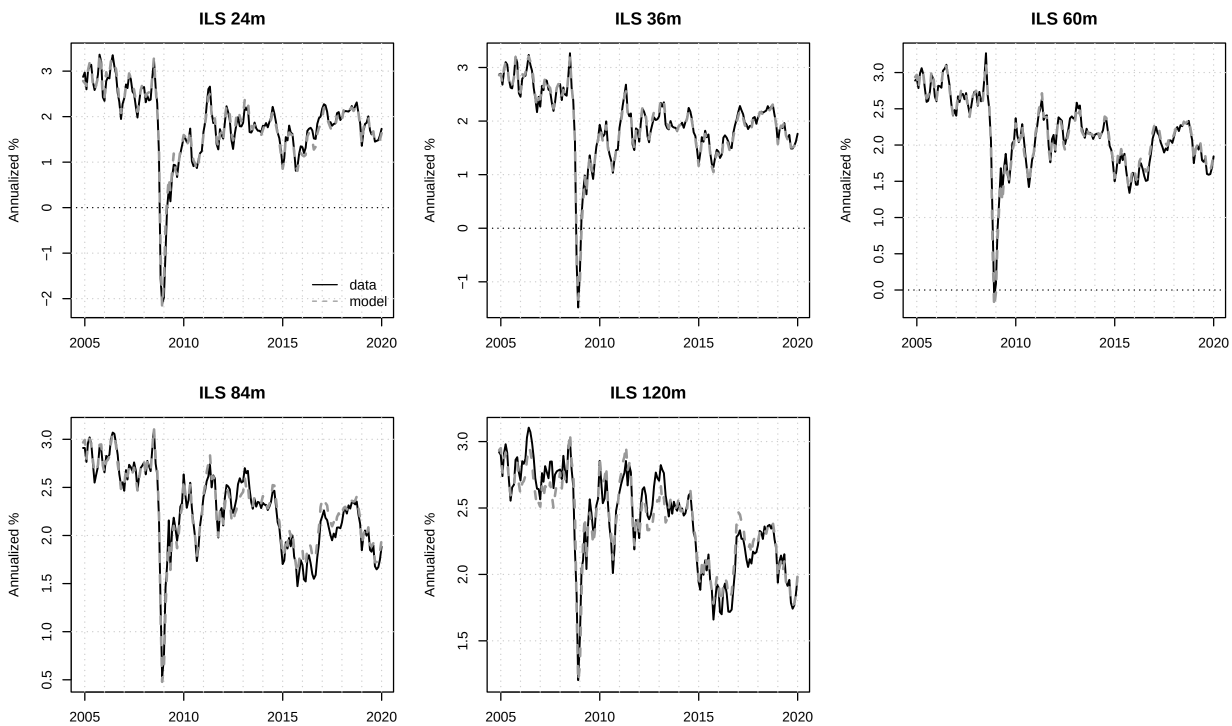


Figure A4: Nominal Treasuries fitted values

The model is estimated with extended Kalman filter. Data range from November 2004 to December 2019. The black solid line presents the observation data used as input for estimation. The grey dashed line presents the fitted values produced through the filtered factors presented on Figure 5.

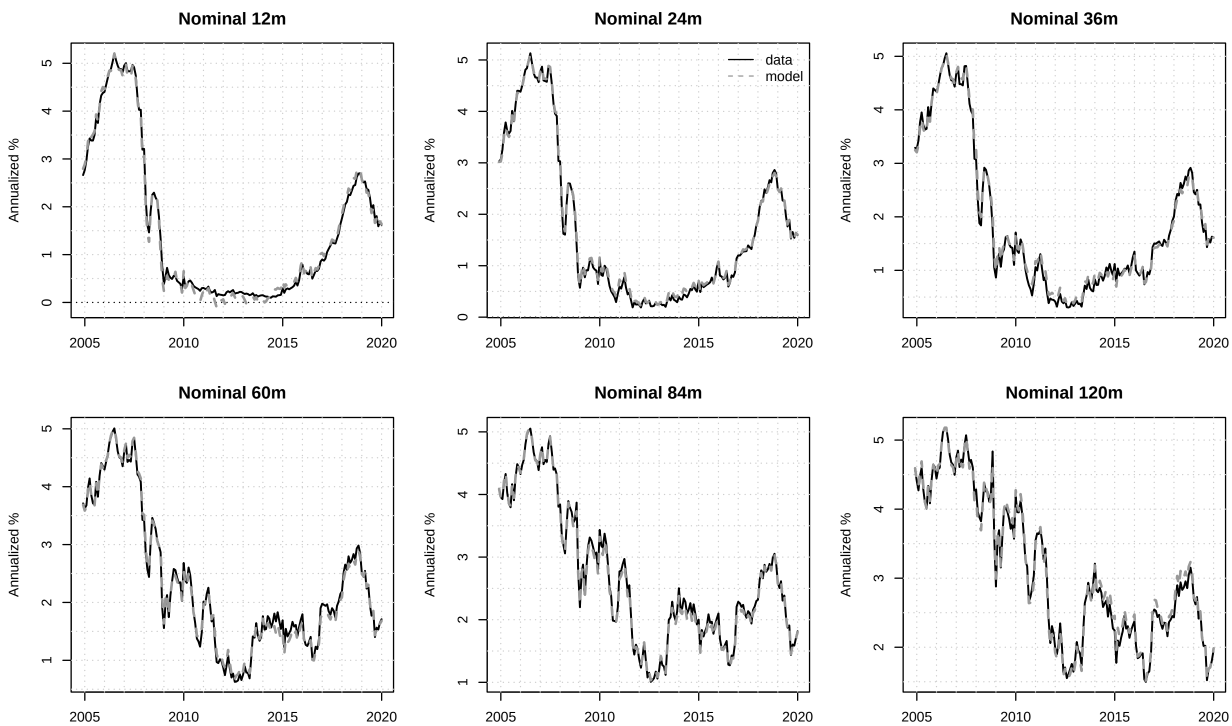


Figure A5: ILSBEI spreads fitted values

The model is estimated with extended Kalman filter. Data range from November 2004 to December 2019. The black solid line presents the observation data used as input for estimation. The grey dashed line presents the fitted values produced through the filtered factors presented on Figure 5.

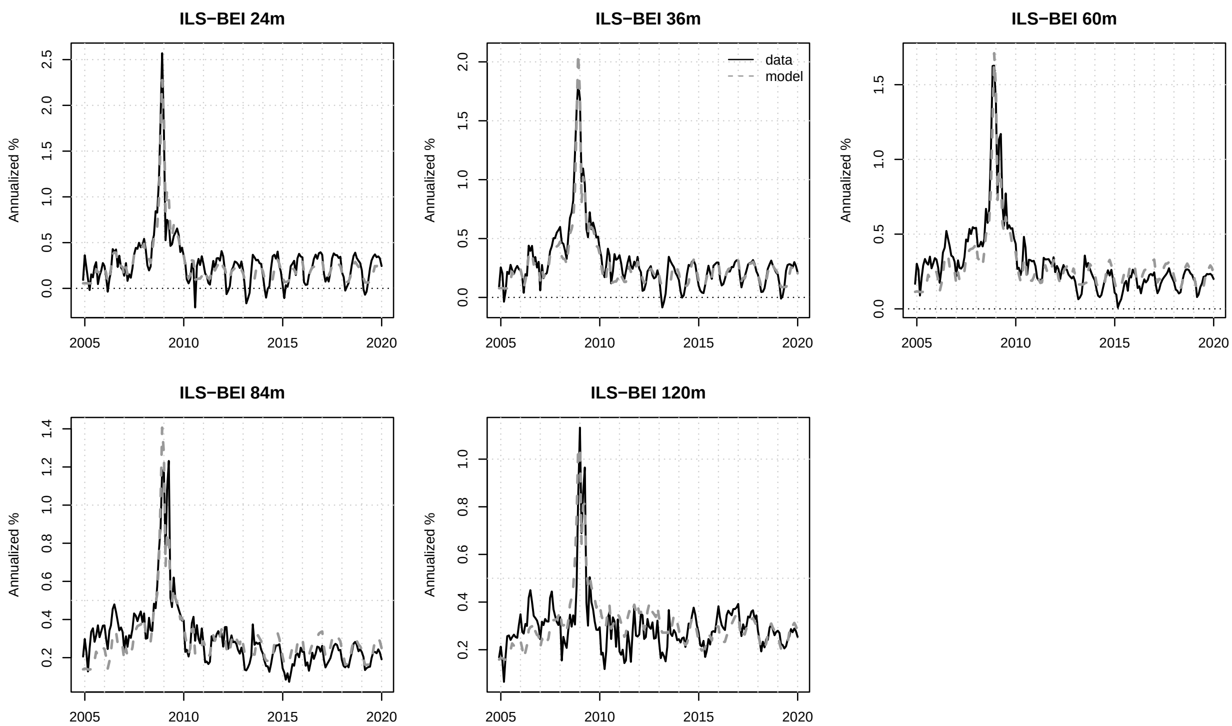


Figure A6: U.S. sovereign CDS spreads fitted values

The model is estimated with extended Kalman filter. Data range from November 2004 to December 2019. The black solid line presents the observation data used as input for estimation. The grey dashed line presents the fitted values produced through the filtered factors presented on Figure 5. The blue zones represent approximate confidence intervals of measurement errors around the filtered CDS estimates. The latter are measured with twice the CDS illiquidity series presented on the bottom panel. The latter is computed as the scale of the inverse aggregated monthly depth (number of Contributors whose contributions were included in the final composite value) of the 5y U.S. sovereign CDS.

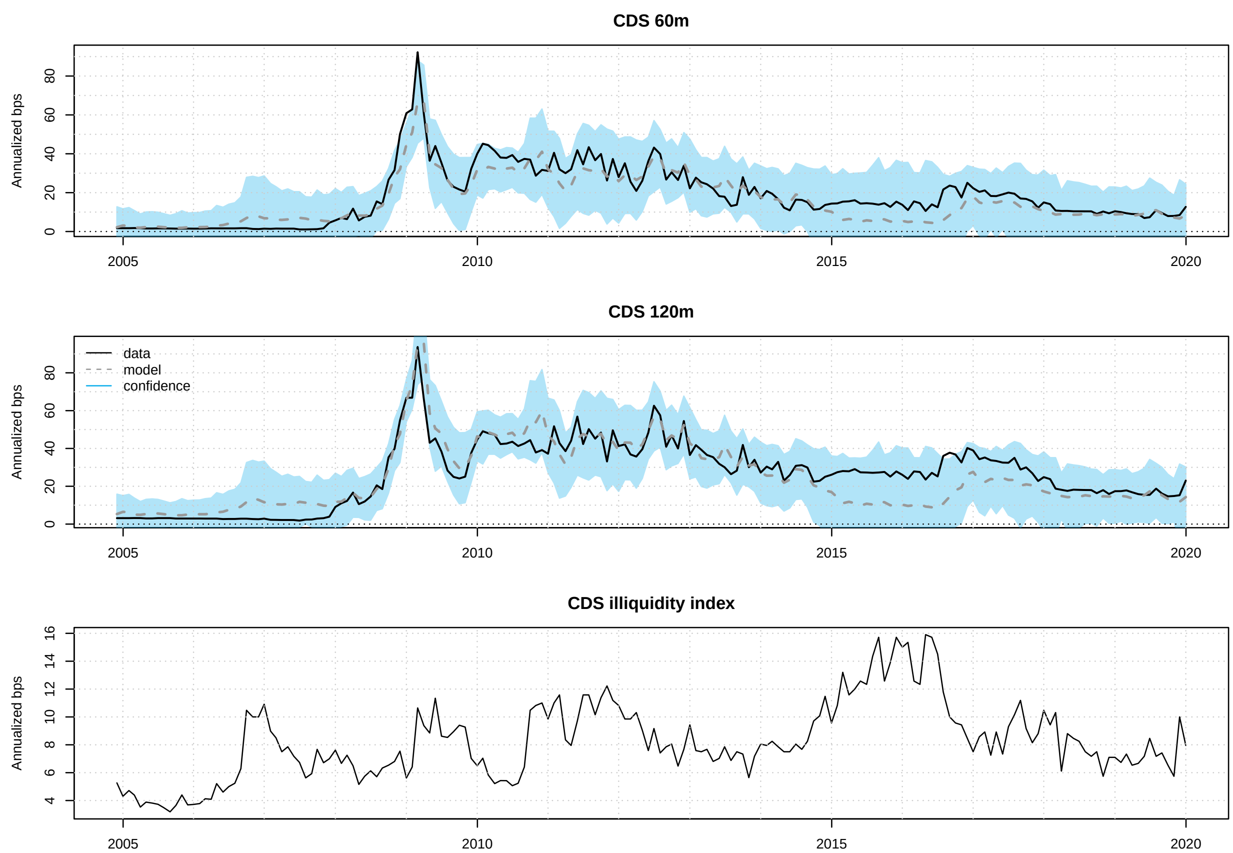


Figure A7: Credit Spreads

This graph presents the nominal and real credit spreads on panel (a) and (b), respectively, for maturities ranging from 1y to 10y. Nominal spreads are obtained taking the difference between observed nominal treasury yields and model-implied riskfree nominal rates. Real spreads are obtained taking the difference between the observed TIPS rates and the model-implied riskless real yields. The model is estimated with extended Kalman filter. Data range from November 2004 to December 2019.